Cylindrical contact homology in dimension 3 via intersection theory and more

Jo Nelson

the IAS and Columbia University

Short Talks, October 1, 2014

What is a contact manifold?

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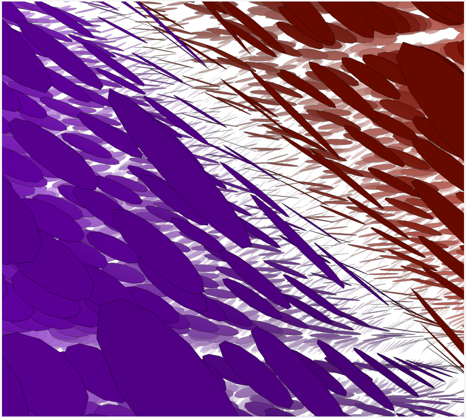

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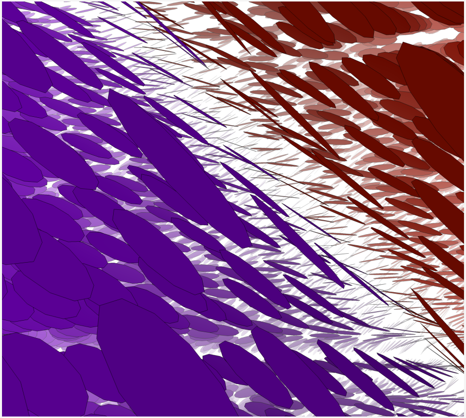

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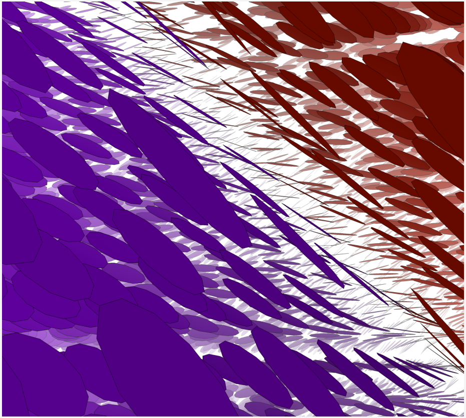

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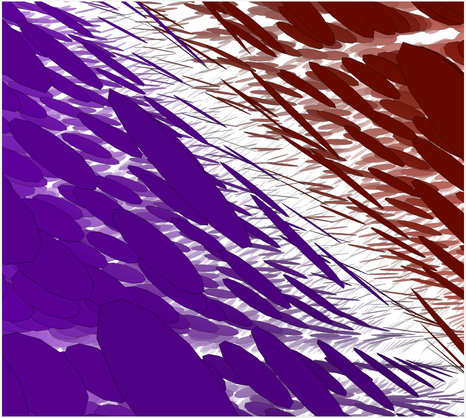

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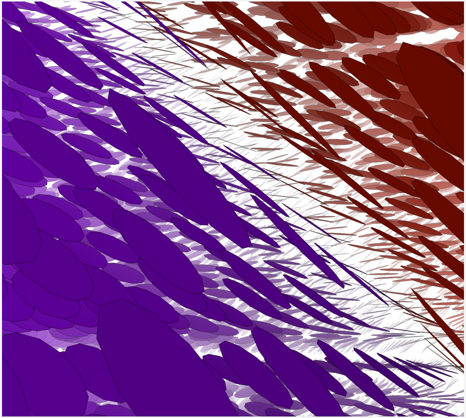

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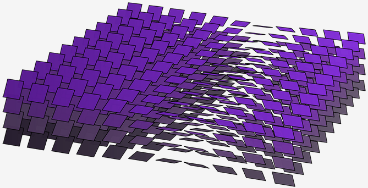

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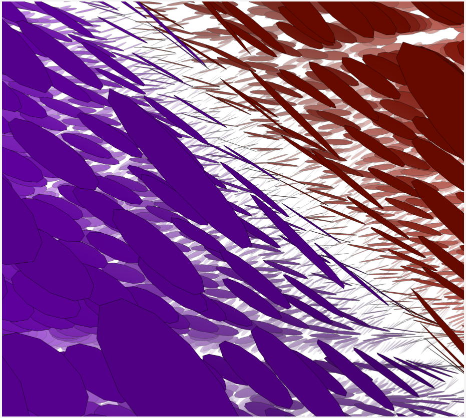

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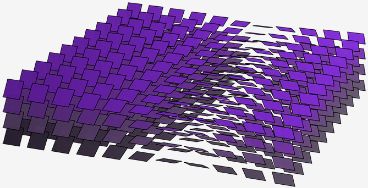

Here: $\alpha = dz - ydx$

Reeb flow

Choose a contact form α.

Definition

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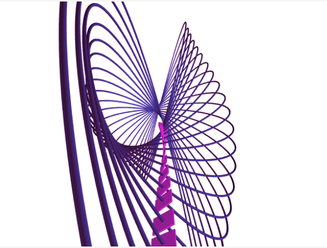
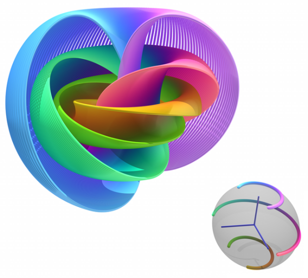


Patrick Massot



http://www.nilesjohnson.net/hopf.html

A dream for a chain complex

Assume: M closed and α nondegenerate

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- $C_*(\alpha) = \{\text{closed Reeb orbits}\} \setminus \{\text{bad Reeb orbits}\}$

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Conjeorem (Eliashberg-Givental-Hofer '00)

Assume a minimal amount of things. Then $(C_*(\alpha), \partial))$ forms a chain complex and $H(C_*(\alpha), \partial)$ is independent of α and $\tilde{J}$.

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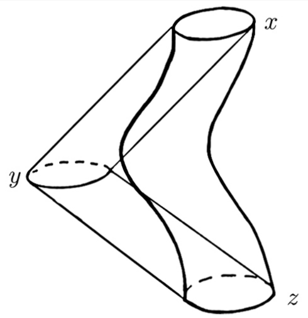

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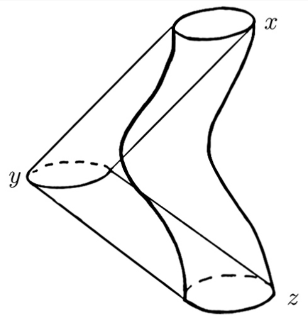

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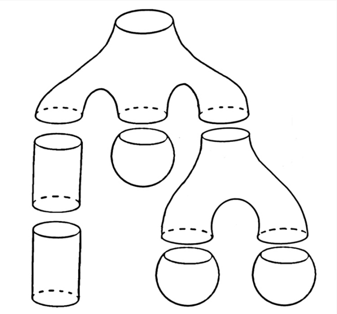

Adding to 2 becomes hard

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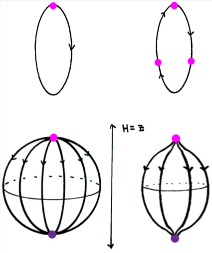

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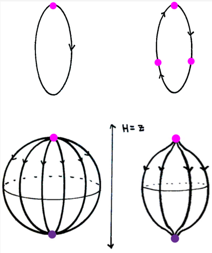

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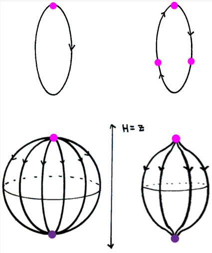

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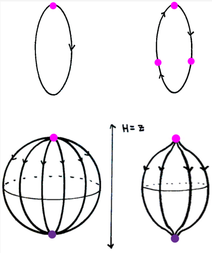

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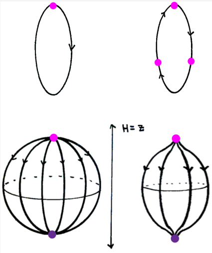

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Theorem (N.)

$\partial^2 = 0$, *invariance under choice of* $\tilde{J}$ *and dynamically separated* α.

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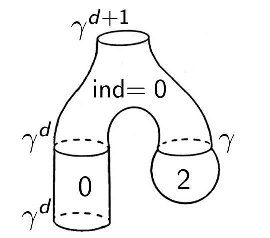

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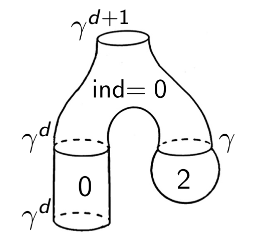

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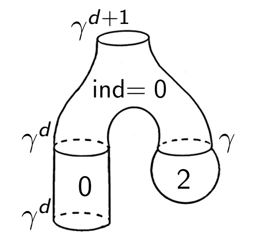

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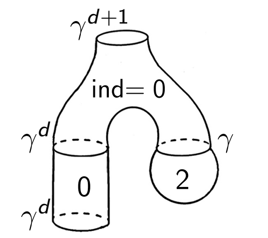

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Theorem (Hutchings-N.)

If (M^3, α) is dynamically convex and every contractible Reeb orbit γ has $\mu_{CZ}(\gamma) = 3$ only if γ is simple then $\partial^2 = 0$.

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- Tensor with $\mathbb{Q}$ to get back CH_*

Theorem (Hutchings-N; in progress)

INVARIANCE! Obtained for dynamically convex (M^3, α) wherein a contractible γ has $\mu_{CZ}(\gamma) = 3$ only if γ is simple.

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- Other dynamical questions involving contact structures

The end!

Thanks!