Cylindrical contact homology as a well-defined homology?

Jo Nelson

Columbia University and the IAS

IAS, September 30, 2013

What is a contact manifold?

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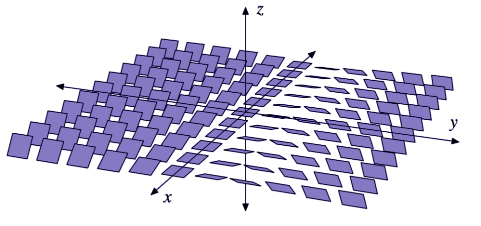

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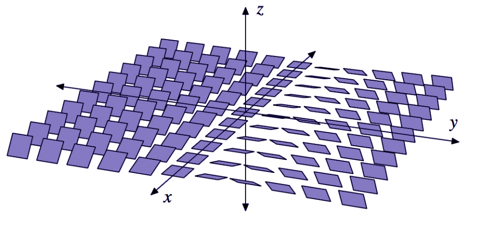

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- $\alpha \wedge (d\alpha)^{n-1}$ is a volume form
- $\Leftrightarrow$ $d\alpha|_\xi$ is nondegenerate

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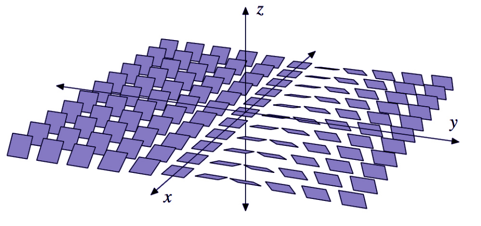

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Above:
$\alpha = dz - ydx$

Reeb flow

Choose a contact form α.

Definition

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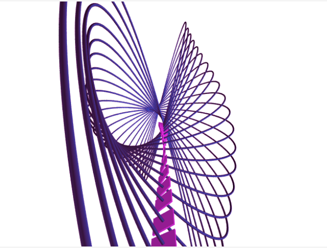
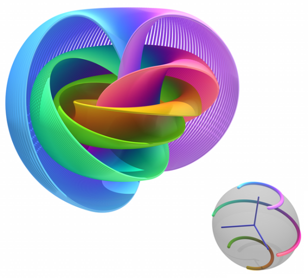


Patrick Massot



http://www.nilesjohnson.net/hopf.html

A dream for a chain complex

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$$\begin{aligned} \mathcal{A}: \quad C^\infty(S^1, M) &\to \mathbb{R}, \\ \gamma &\mapsto \int_\gamma \alpha. \end{aligned}$$

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- $C_*(\alpha) = \{\text{closed Reeb orbits}\} \setminus \{\text{bad Reeb orbits}\}$

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- Hope this is independent of our choices.

Conjeorem (Eliashberg-Givental-Hofer '00)

Assume a minimal amount of things. Then $(C_*(\alpha), \partial))$ forms a chain complex and $H(C_*(\alpha), \partial)$ is independent of α and $\tilde{J}$.

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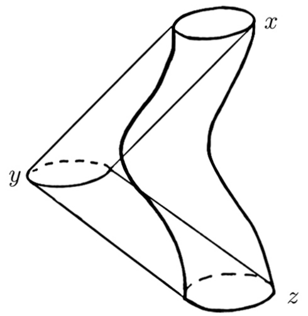

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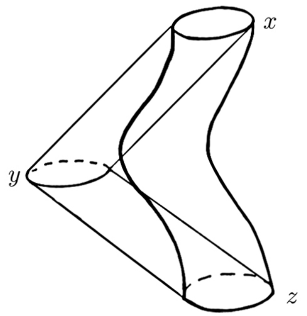

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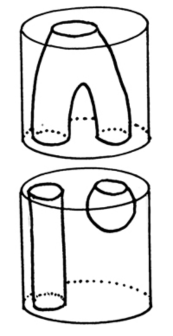

Adding to 2 becomes hard

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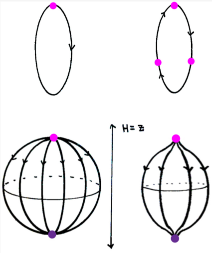

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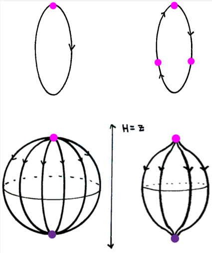

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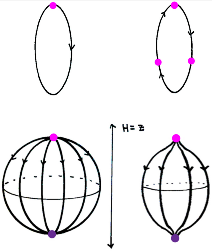

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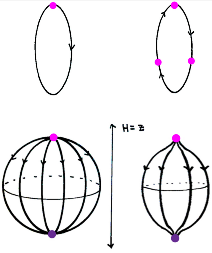

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(ii) $\mu_{CZ}(\gamma^k) = \mu_{CZ}(\gamma^{k-1}) + 4, \gamma^k$ is the k-th iterate of a simple orbit γ.

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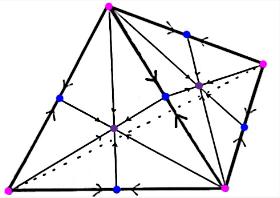

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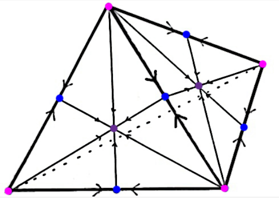

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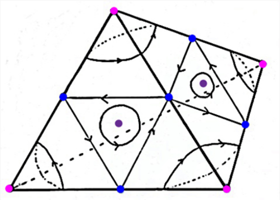

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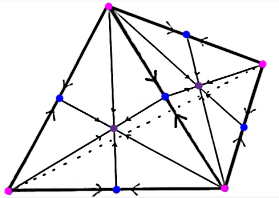

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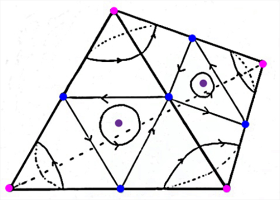

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Reeb orbits which generate chain complex correspond to presentation of S^3/Γ as a Seifert fiber space!

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- Other dynamical questions involving contact structures

Thanks!