p-adic L-functions and Iwasawa main conjectures

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Institute for Advanced Study

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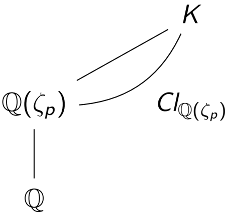

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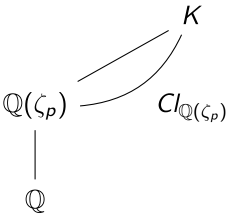

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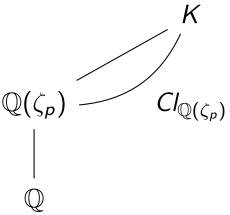

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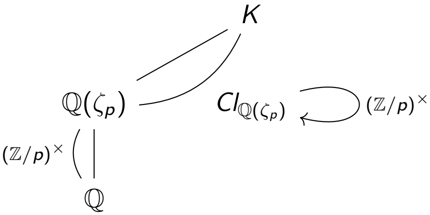

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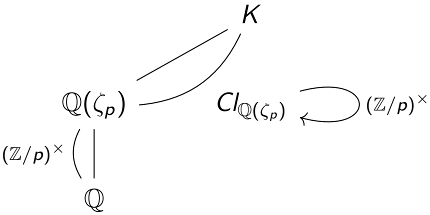

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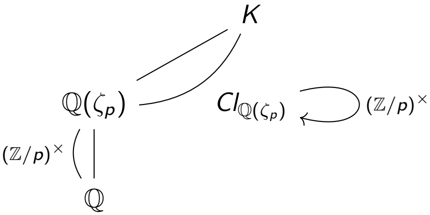

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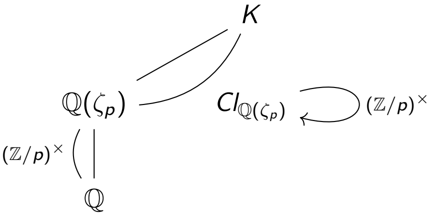

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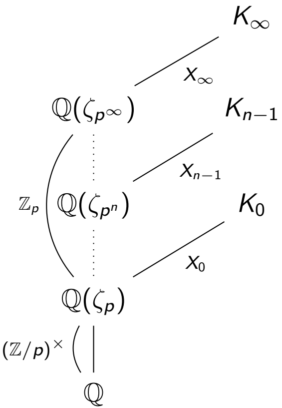

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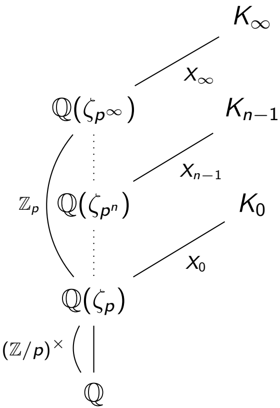

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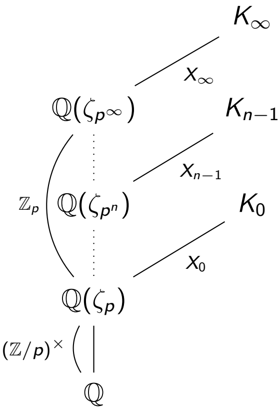

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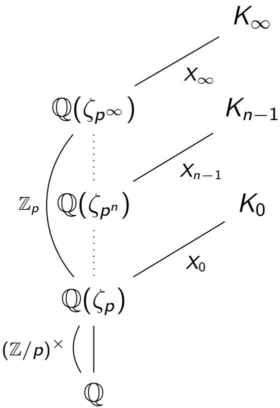

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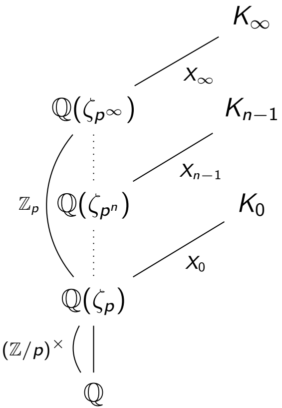

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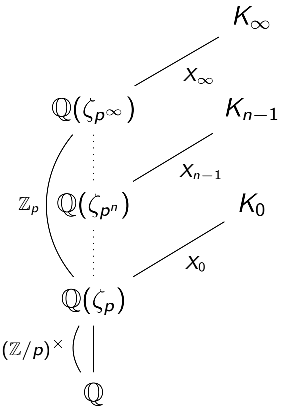

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The Iwasawa main conjecture can be viewed as a generalized class number formula, and **the p-adic L-function** is the object appearing on the analytic side. This picture generalizes (elliptic curves, automorphic Galois representations ...)

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"Theorem" (L)

- *If π is ordinary, there exists the one-variable p-adic L-function $\mathcal{L}_{\pi, p\text{-adic}}$.*
- *For an n-variable Hida eigen-family $\mathcal{C}$, there exists the $(n+1)$-variable p-adic L-function $\mathcal{L}_{\mathcal{C}}$.*

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- selecting nice sections at p and ∞ for the Siegel Eisenstein series (desired p-adic congruences, nonvanishing of local zeta integrals ...),
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The construction relies on the **doubling method**, developed by Garrett, Piatetski-Shapiro–Rallis, Shimura ... It roughly says

$$\left\langle E^{\mathrm{Sieg}}(s,\chi)\big|_{\mathrm{Sp}(2n)\times\mathrm{Sp}(2n)}, \varphi\otimes\overline{\varphi}\right\rangle \sim L(s+\frac{1}{2},\pi\times\chi).$$

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Our construction involves

- selecting nice sections at p and ∞ for the Siegel Eisenstein series (desired p-adic congruences, nonvanishing of local zeta integrals ...),
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- computing the zeta integrals at p,

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$$\mathrm{Char}_{\mathbb{I}_{\mathcal{C}}[[T]]}\left(X^{S}_{\mathbb{Q}_\infty}(\mathcal{C})\right) = \left(\mathcal{L}_{\mathcal{C}}\right).$$

There is a common strategy for attempting to prove one divisibility in the aforementioned theorems and conjectures,

$$L\text{-value} \mid \text{size of the Selmer group},$$

$$p\text{-adic } L\text{-function} \mid \text{characteristic ideal of the Selmer group}.$$

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In the case $\pi \subset \mathcal{A}_0(\mathrm{Sp}(2n, \mathbb{Q}) \backslash \mathrm{Sp}(2n, \mathbb{A}))$, after constructing the Klingen Eisenstein family satisfying the first property (still by doubling method), I am interested in computing the corresponding non-degenerate Fourier coefficients, and finding how far I can go in the general setting as well as if there are some special cases for which the second property can be verified.

In general,

$$\left\langle \varphi,\, \theta_{n+1}(\phi)(\cdot,1)E^{\mathrm{Sieg}}(s) \right\rangle,$$

where $\varphi \in \pi$, $\phi \in \mathcal{S}\left(M_{n,n+1}(\mathbb{A})\right)$.

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$$\begin{array}{ccc} \theta_{n+1}(\phi)(\cdot,1) & & E^{\mathrm{Sieg}}(s)|_{s=k-n-1} \\ \widetilde{\mathrm{Sp}}(2n) & \times & \widetilde{\mathrm{Sp}}(2n) \\ & \Big| & \\ & \mathrm{Sp}(2n) & \\ & \varphi \in \pi & \end{array}$$

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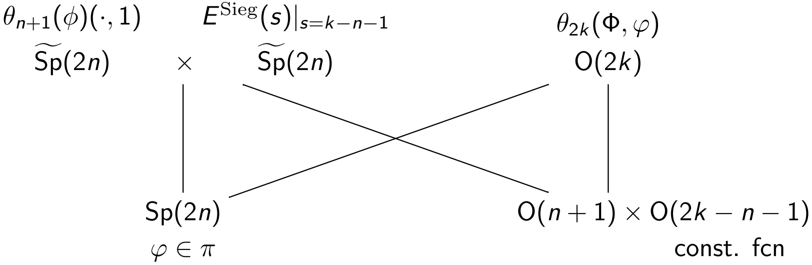

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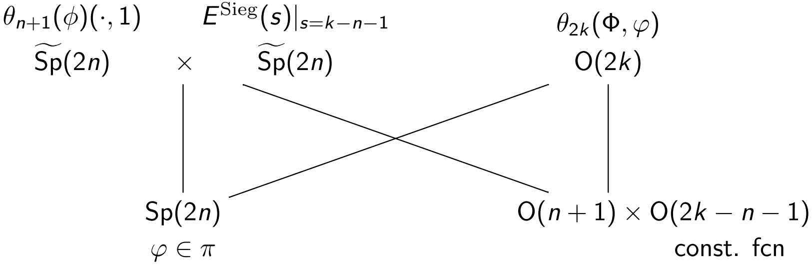

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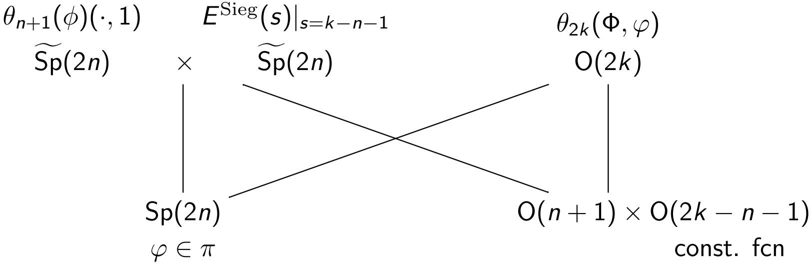

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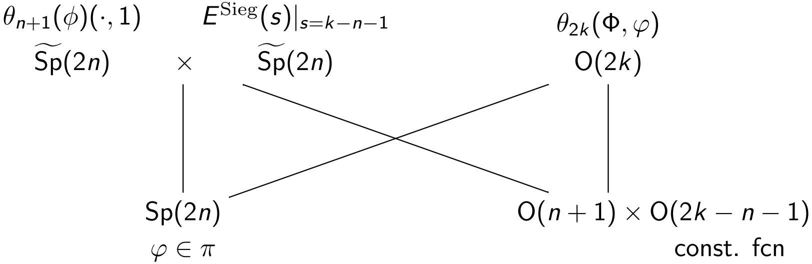

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- understand $\left\langle \varphi,\, \theta_{n+1}(\phi)(\cdot,1)E^{\mathrm{Sieg}}(s)\right\rangle\Big|_{s=k-n-1}$,
- relate the change of indices of Fourier coefficients to the translation by $\mathrm{O}(2k)$ on Φ,
- say something about the function

$$h \mapsto \int_{\mathrm{O}(2k-n-1,\mathbb{Q})\backslash\,\mathrm{O}(2k-n-1,\mathbb{A})} \theta_{2k}(\Phi,\varphi)(xh)\,dx.$$

Thank you!