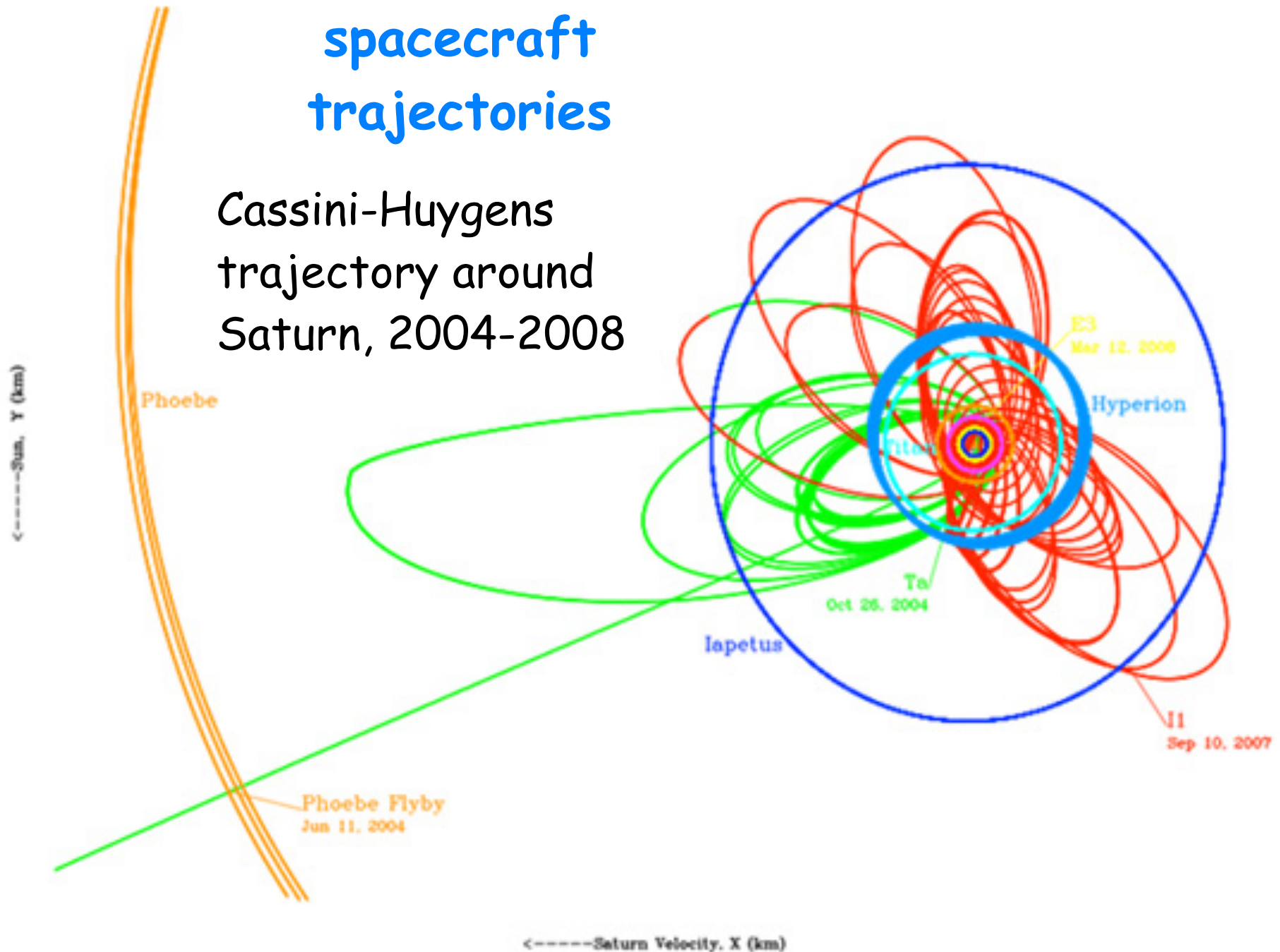


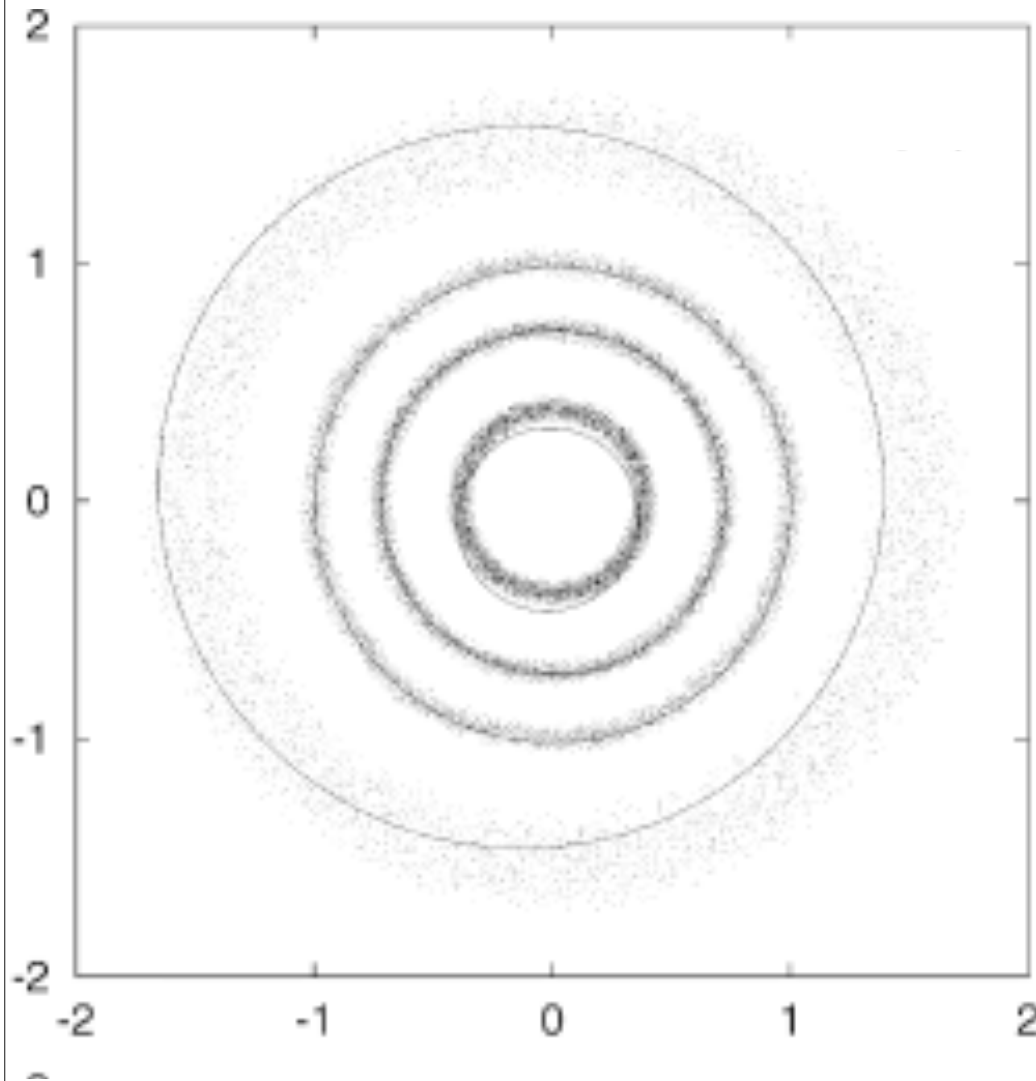
# ***Geometric methods for orbit integration***

# spacecraft trajectories

Cassini-Huygens  
trajectory around  
Saturn, 2004-2008



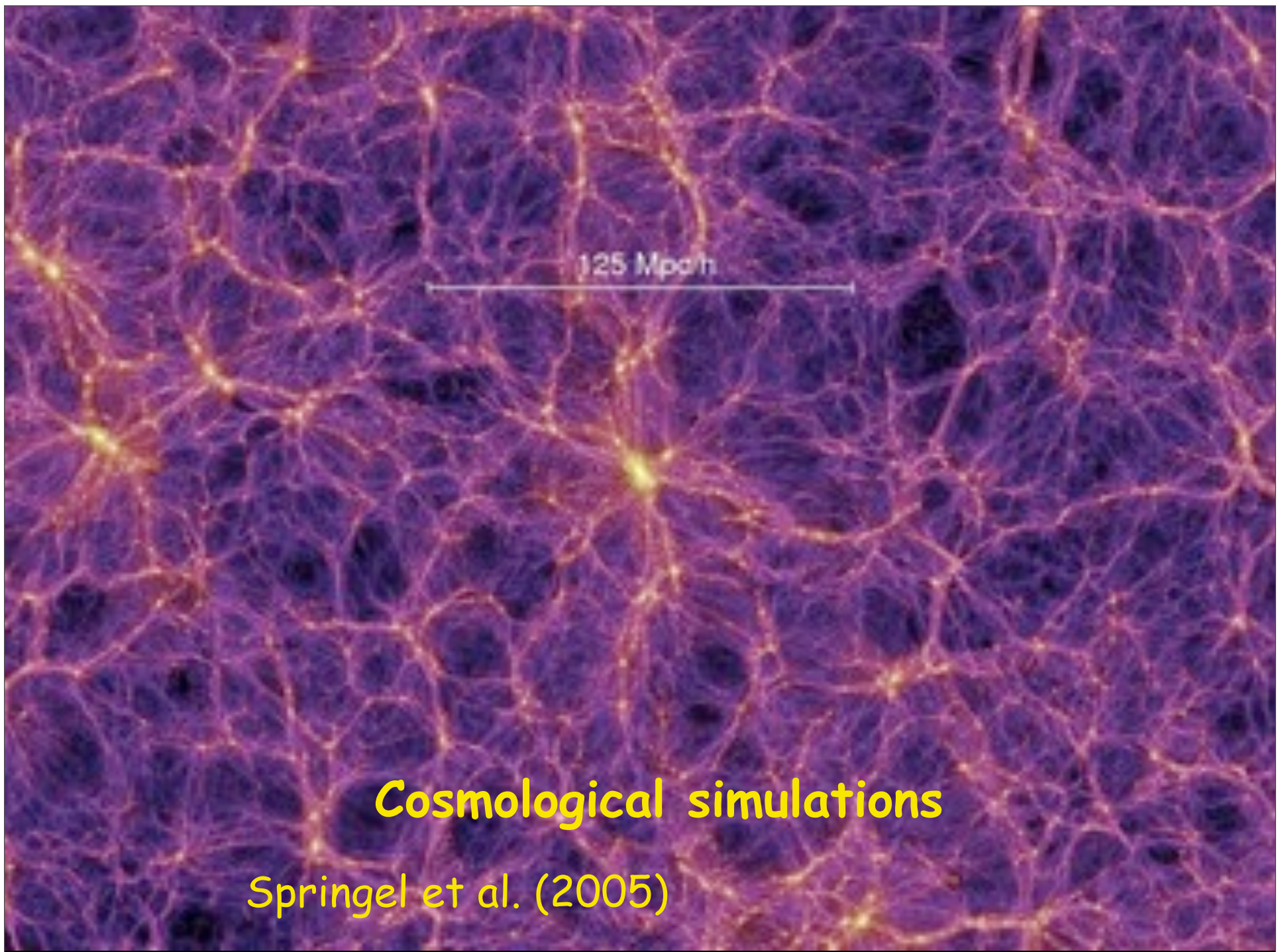
## Planetary orbits



lines = current orbits of the  
four inner planets

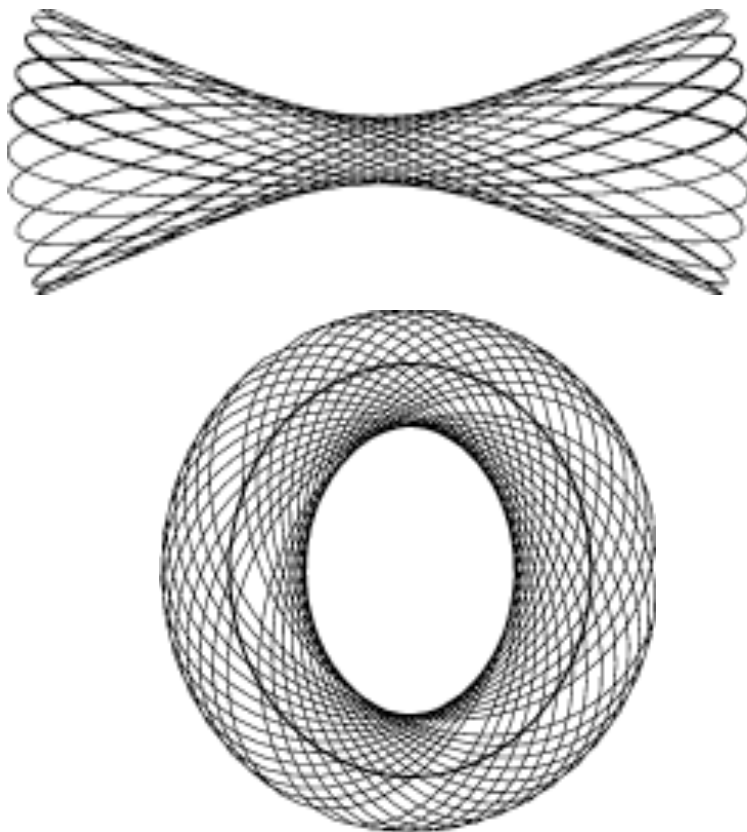
dots = orbits of the inner  
planets over 50,000 years, 4.5  
Gyr in the future

Ito & Tanikawa (2002)

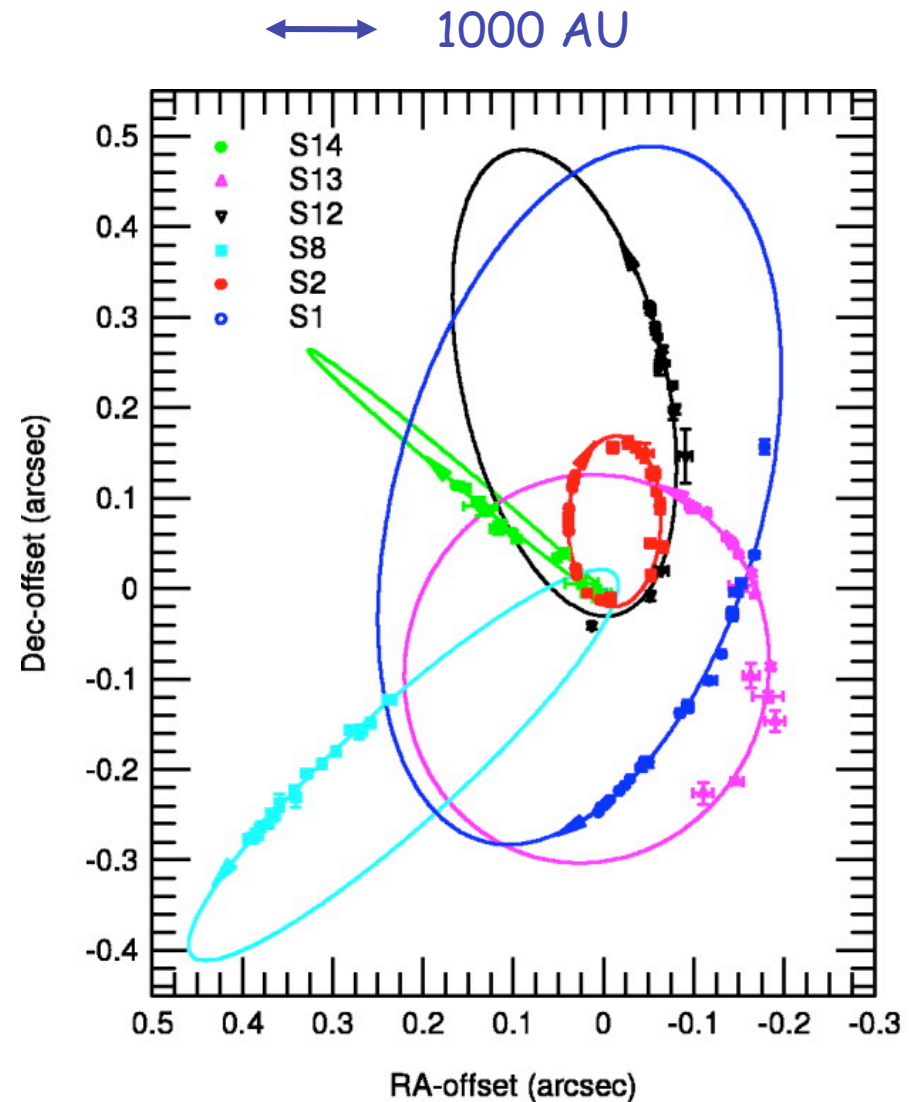




# Galactic dynamics



box and tube orbits in a  
galactic potential



orbits of stars near the  
Galactic center

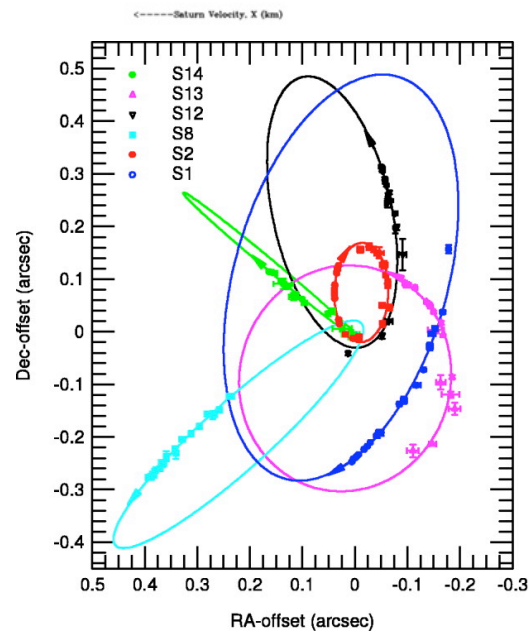
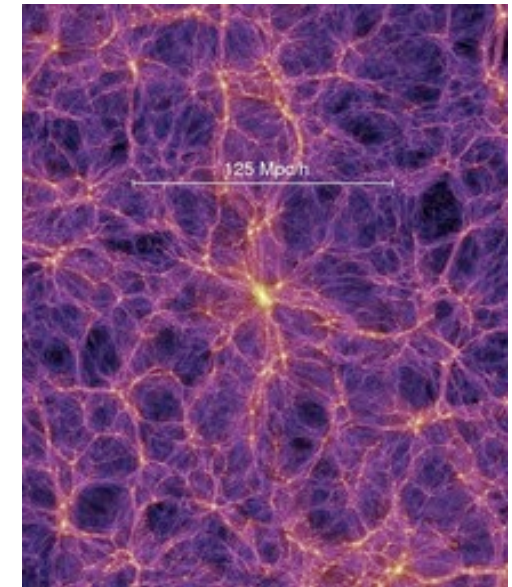
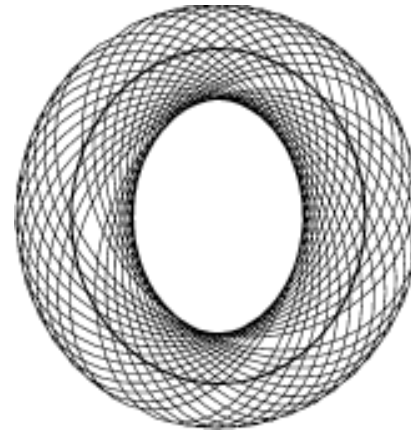
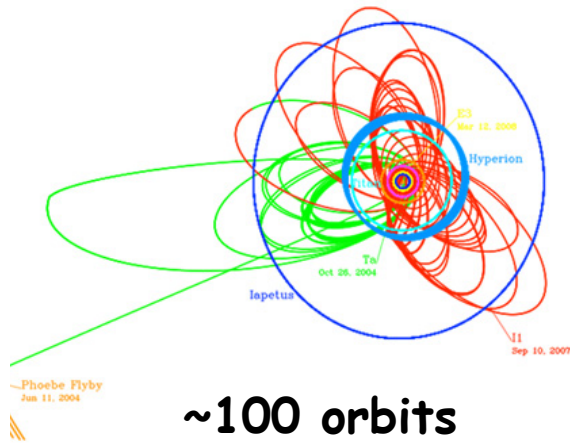
Eisenhauer et al. (2005)

# Large Hadron Collider

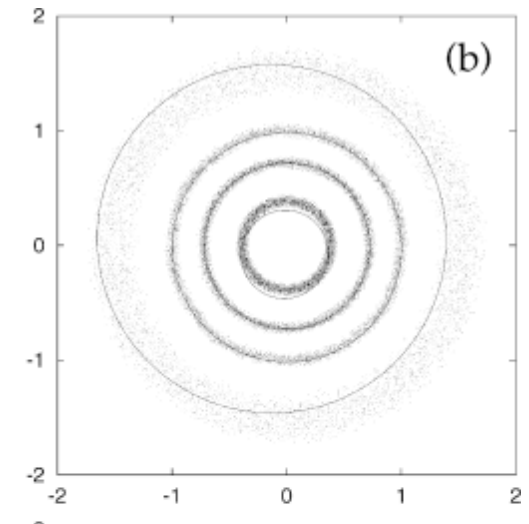
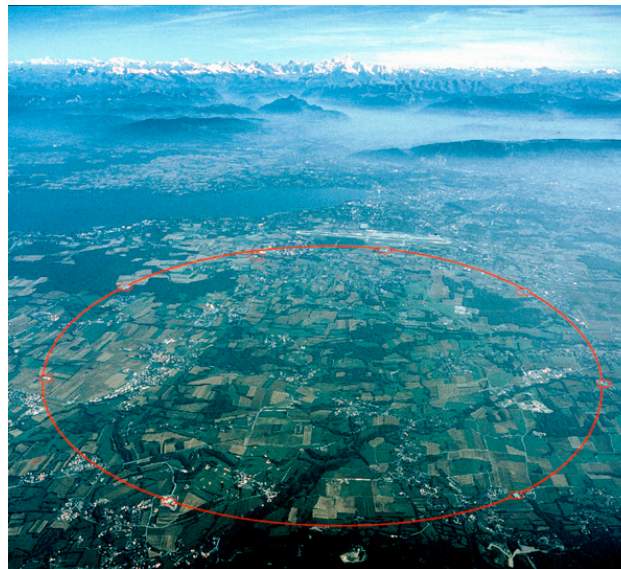


Wednesday, November 9, 2011





~10<sup>9</sup> orbits



Consider following a particle in the force field of a point mass.  
Set  $G=M=1$  for simplicity. Equations of motion read

$$\dot{\mathbf{r}} = \mathbf{v} \quad ; \quad \dot{\mathbf{v}} = \mathbf{F}(\mathbf{r}) = -\frac{\hat{\mathbf{r}}}{r^2}$$

Examine three integration methods with timestep  $h$ :

$$\mathbf{r}_{n+1} = \mathbf{r}_n + h\mathbf{v}_n \quad ; \quad \mathbf{v}_{n+1} = \mathbf{v}_n + h\mathbf{F}(\mathbf{r}_n)$$

1. Euler's method

$$\mathbf{r}_{n+1} = \mathbf{r}_n + h\mathbf{v}_n \quad ; \quad \mathbf{v}_{n+1} = \mathbf{v}_n + h\mathbf{F}(\mathbf{r}_{n+1})$$

2. modified Euler's

$$\mathbf{r}' = \mathbf{r}_n + \frac{1}{2}h\mathbf{v}_n \quad ; \quad \mathbf{v}_{n+1} = \mathbf{v}_n + h\mathbf{F}(\mathbf{r}') \quad ; \quad \mathbf{r}_{n+1} = \mathbf{r}' + \frac{1}{2}h\mathbf{v}_{n+1}$$

3. leapfrog

4. Runge-Kutta method

Euler methods are **first-order**; leapfrog is **second-order**; Runge-Kutta is **fourth order**

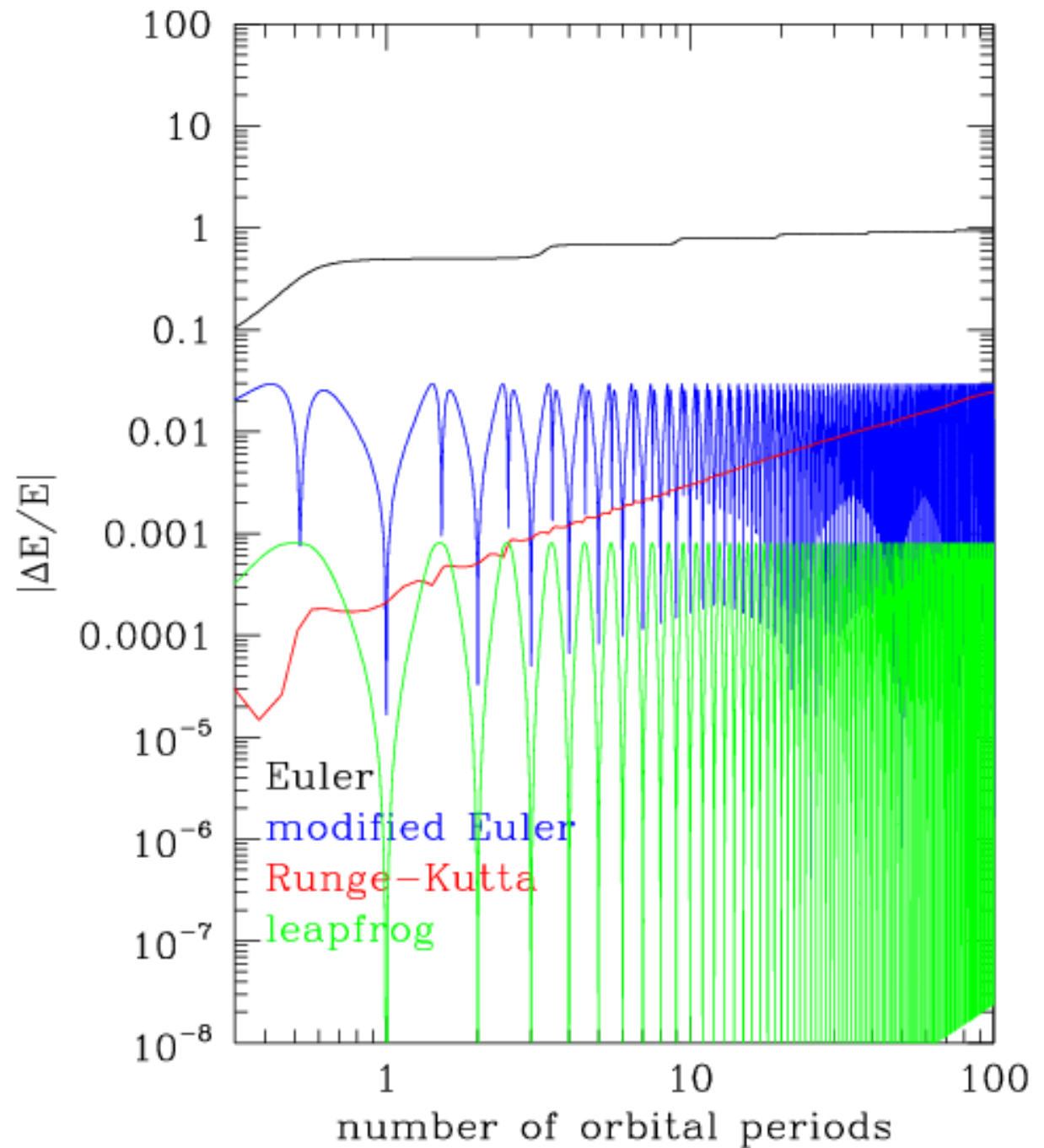
Use equal number of force evaluations per orbit for each method  
(rather than equal timesteps)



eccentricity = 0.2

200 steps per  
orbit

plot shows  
fractional energy  
error  $|\Delta E/E|$





A **geometric integration algorithm** is a numerical integration algorithm that exactly preserves some geometric property of the original set of differential equations

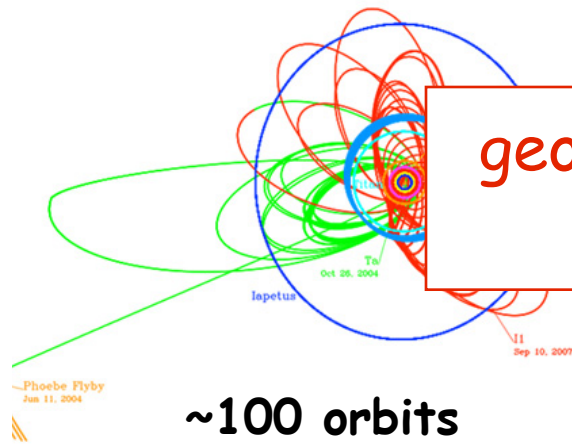
### Volume-conserving algorithms:

- conserve phase-space volume, i.e. satisfy Liouville's theorem
- appropriate for Hamiltonian systems
- e.g. modified Euler, leapfrog but **not** Runge-Kutta





The motivation for geometric integration algorithms is that **preserving the phase-space geometry of the flow determined by the real dynamical system is more important than minimizing the one-step error**



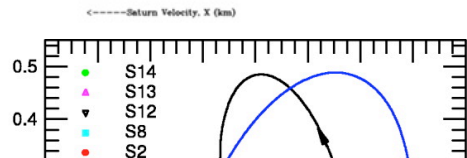
geometric integrators not really necessary

~100-1000 orbits

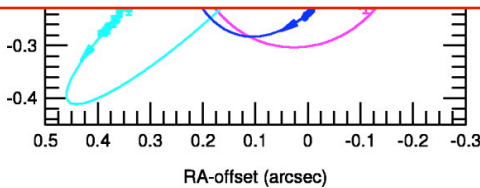
~10<sup>9</sup> orbits



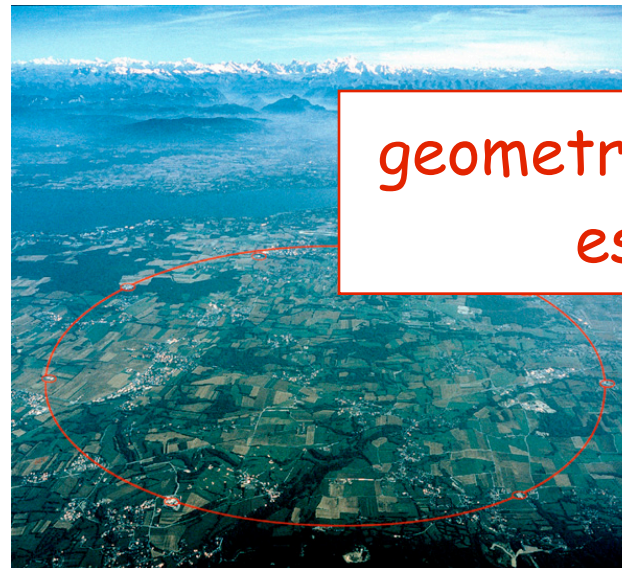
~100-1000 orbits



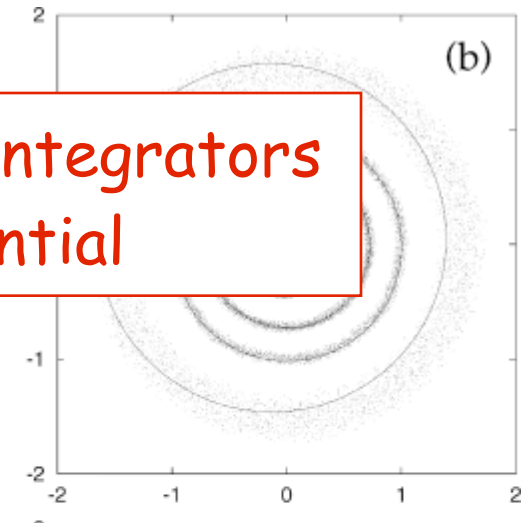
geometric integrators helpful



~10<sup>6</sup> orbits



geometric integrators essential



~10<sup>10</sup> orbits



## Energy-conserving algorithms:

- conserve energy, i.e. restrict the system to a surface of constant energy in phase space
- appropriate for systems with time-independent Hamiltonians, e.g. motion in a fixed potential
- does **not** include modified Euler, leapfrog, Runge-Kutta

## Time-reversible algorithms:

- integrate forward in time for N steps, reverse all velocities, integrate backward in time for N steps, reverse velocities, and the system is back where it started
- appropriate for time-reversible systems, e.g. gravitational N-body problem
- includes leapfrog but not modified Euler or Runge-Kutta

$$\mathbf{r}' = \mathbf{r}_n + \frac{1}{2}h\mathbf{v}_n ; \mathbf{v}_{n+1} = \mathbf{v}_n + h\mathbf{F}(\mathbf{r}') ; \mathbf{r}_{n+1} = \mathbf{r}' + \frac{1}{2}h\mathbf{v}_{n+1}$$

## Symplectic algorithms:

- if the dynamical system is described by a Hamiltonian  $H(q,p)$  then

$$\frac{d\mathbf{p}}{dt} = -\frac{\partial H}{\partial \mathbf{q}} \quad ; \quad \frac{d\mathbf{q}}{dt} = \frac{\partial H}{\partial \mathbf{p}}.$$

- if  $y(t)=[q(t),p(t)]$  then the flow from  $y(t_0)$  to  $y(t_1)$  is a **symplectic** or canonical map
- an integration method is symplectic if the formula for advancing by one timestep

$$y_{n+1} = y_n + g(t_n, y_n, h)$$

is also a symplectic map

- for one-dimensional systems symplectic = volume-conserving (actually area-conserving)
- for systems of more than one dimension symplectic is more general
- modified Euler and leapfrog are symplectic





# Symplectic algorithms for separable Hamiltonians

If the dynamical system is described by a Hamiltonian of the form

$$H = H_D + H_K, \quad H_D = \frac{1}{2}\mathbf{v}^2, \quad H_K = \Phi(\mathbf{x}, t)$$

then the *drift* operator  $\mathbf{D}_h$  advances the orbit for time  $h$  under the influence of  $H_D$  alone:

$$\mathbf{D}_h : (\mathbf{x}, \mathbf{v}) \rightarrow (\mathbf{x} + \mathbf{v}h, \mathbf{v}).$$

Similarly the *kick* operator  $\mathbf{K}_h$  advances the orbit under the influence of  $H_K$  alone:

$$\mathbf{K}_h : (\mathbf{x}, \mathbf{v}) \rightarrow (\mathbf{x}, \mathbf{v} - h\nabla\Phi(\mathbf{x})).$$

The modified Euler operator is

$$\mathbf{K}_h\mathbf{D}_h : (\mathbf{x}, \mathbf{v}) \rightarrow [\mathbf{x} + \mathbf{v}h, \mathbf{v} - \nabla\Phi(\mathbf{x} + \mathbf{v}h)].$$

Similarly the leapfrog operator is

$$\mathbf{D}_{h/2}\mathbf{K}_h\mathbf{D}_{h/2} \quad \text{or} \quad \mathbf{K}_{h/2}\mathbf{D}_h\mathbf{K}_{h/2}$$

# Symplectic algorithms

An  $n^{\text{th}}$ -order symplectic integration algorithm for a Hamiltonian  $H$  gives the exact trajectory for a nearby Hamiltonian  $H+H_{\text{error}}$  where  $H_{\text{error}}$  is  $O(h^n)$

This result changes the study of numerical integration from numerical analysis (boring) to dynamics (interesting)

e.g., integrating pendulum Hamiltonian  $H=v^2/2+C \cos(x)$  with modified Euler method gives

$$x' = x + hv \quad , \quad v' = v + Ch \sin(x')$$

set  $hv=y$

$$x' = x + y \quad , \quad y' = y + K \sin(x') \quad K=Ch^2$$

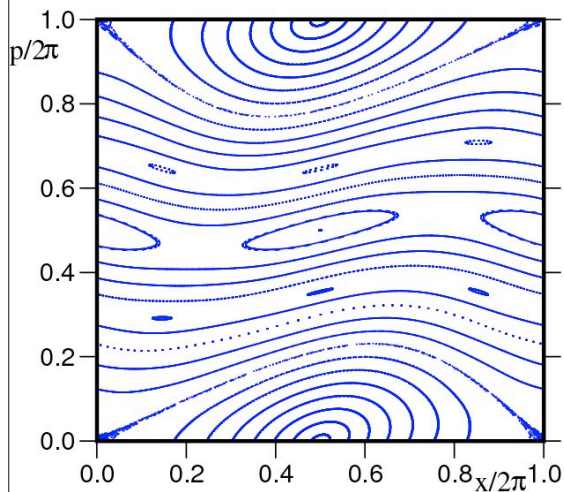
This is the Chirikov-Taylor or standard or kicked pendulum map

# Symplectic algorithms

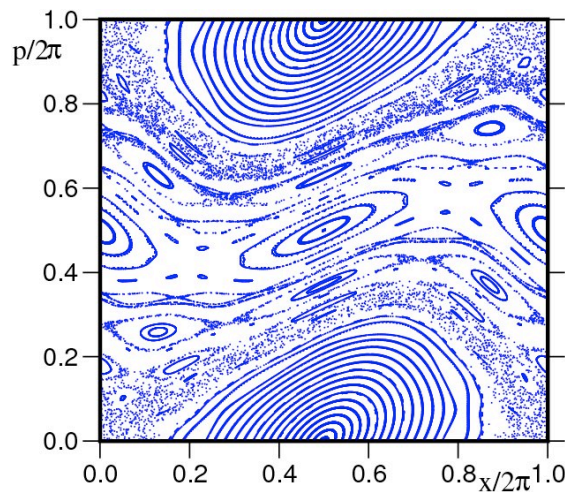
$$x' = x + y \quad . \quad y' = y + K \sin(x') \quad K = Ch^2$$

This is the Chirikov-Taylor or standard or kicked pendulum map

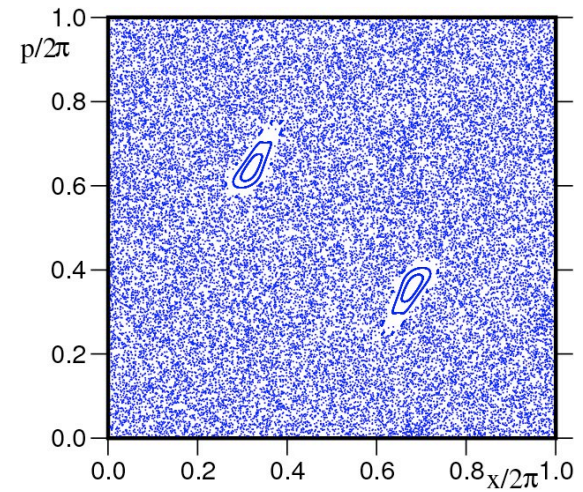
$K=0.5$



$K=0.971635$



$K=5$



# Geometric integrators for cosmology

The Lagrangian for a gravitational N-body system is

$$L = \frac{1}{2} \sum_i m_i \dot{\mathbf{r}}_i^2(t) - G \sum_{j>i} \frac{m_i m_j}{|\mathbf{r}_i - \mathbf{r}_j|}$$

Introduce comoving coordinates  $\mathbf{x}$  by  $\mathbf{r} = a(t)\mathbf{x}$

$$L = \frac{1}{2} \sum_i m_i (\dot{a}\mathbf{x}_i + a\dot{\mathbf{x}}_i)^2 - \frac{G}{a} \sum_{j>i} \frac{m_i m_j}{|\mathbf{x}_i - \mathbf{x}_j|}$$

Momentum is  $\mathbf{p}_i = \partial L / \partial \dot{\mathbf{x}}_i = m_i(a\dot{a}\mathbf{x}_i + a^2\dot{\mathbf{x}}_i)$

Hamiltonian is  $H(\mathbf{q}, \mathbf{p}, t) = \sum \mathbf{p}_i \cdot \dot{\mathbf{q}}_i - L = H_A(\mathbf{q}, \mathbf{p}, t) + H_B(\mathbf{q}, \mathbf{p}, t)$

with

$$H_A = \sum_i \frac{\mathbf{p}_i^2}{2m_i a^2(t)}, \quad H_B = - \sum_{i>j} \frac{G m_i m_j}{a(t) |\mathbf{x}_i - \mathbf{x}_j|}.$$

Drift and kick operators correspond to motion under  $H_A$  and  $H_B$ :

$$\mathbf{x}'_i = \mathbf{x}_i + \frac{\mathbf{p}_i}{m_i} \int_t^{t+h} \frac{dt}{a^2(t)}, \quad \mathbf{p}'_i = \mathbf{p}_i - G \sum_j \frac{\mathbf{x}_i - \mathbf{x}_j}{|\mathbf{x}_i - \mathbf{x}_j|^3} \int_t^{t+h} \frac{dt}{a(t)}$$



# Geometric integrators for planetary systems

To follow motion in the general potential  $\Phi(\mathbf{r}, t)$  we may use the Hamiltonian splitting

$$H(\mathbf{q}, \mathbf{p}, t) = H_A(\mathbf{q}, \mathbf{p}, t) + H_B(\mathbf{q}, \mathbf{p}, t)$$

with

$$H_A = \frac{1}{2}p^2, \quad H_B = \Phi(\mathbf{q}, t)$$

Then integrate using the leapfrog operator  $A_{h/2}B_hA_{h/2}$ .

Motion of a test particle in a planetary system is described by

$$\Phi(\mathbf{r}, t) = -\frac{GM_*}{r} - \sum_j \frac{Gm_j}{|\mathbf{r} - \mathbf{r}_j|}$$

In this case a much better split is

$$H_A = \frac{1}{2}p^2 - \frac{GM_*}{r}, \quad H_B = \sum_j \frac{Gm_j}{|\mathbf{r}_i - \mathbf{r}_j|}$$

The workhorse for long orbit integrations in planetary systems is the **mixed-variable symplectic integrator** (Wisdom & Holman 1991)

$$H(\mathbf{r}, \mathbf{p}) = H_A + H_B,$$

with

$$H_A = \frac{1}{2}p^2 - \frac{GM_\star}{r}, \quad H_B = \sum_j \frac{Gm_j}{|\mathbf{r}_i - \mathbf{r}_j|}$$

and the operator  **$A_{h/2}B_hA_{h/2}$** .

- motion under  $H_A$  is analytic (Keplerian motion) and motion under  $H_B$  is also analytic (impulsive kicks from the planets)
- this is a geometric integrator (symplectic and time-reversible)
- errors smaller than leapfrog by of order  $m_{\text{planet}}/M_\star \sim 10^{-4}$

The workhorse for long orbit integrations in planetary systems is the **mixed-variable symplectic integrator** (Wisdom & Holman 1991)

$$H(\mathbf{r}, \mathbf{p}) = H_A + H_B,$$

with

$$H_A = \frac{1}{2}p^2 - \frac{GM_\star}{r}, \quad H_B = \sum_j \frac{Gm_j}{|\mathbf{r}_i - \mathbf{r}_j|}$$

and the operator  **$A_{h/2}B_hA_{h/2}$** .

This integrator exactly follows the motion in a Hamiltonian  $H+H_{\text{error}}$  where  $H_{\text{error}}$  is  $O(h^2)$  and oscillatory. Dominant errors can be reduced to  $O(m_{\text{planet}}/M^\star)^2$  by ensuring that the action associated with  $H_{\text{error}}$  is zero

This can be done by the “warmup” starting procedure (Saha & Tremaine 1992)

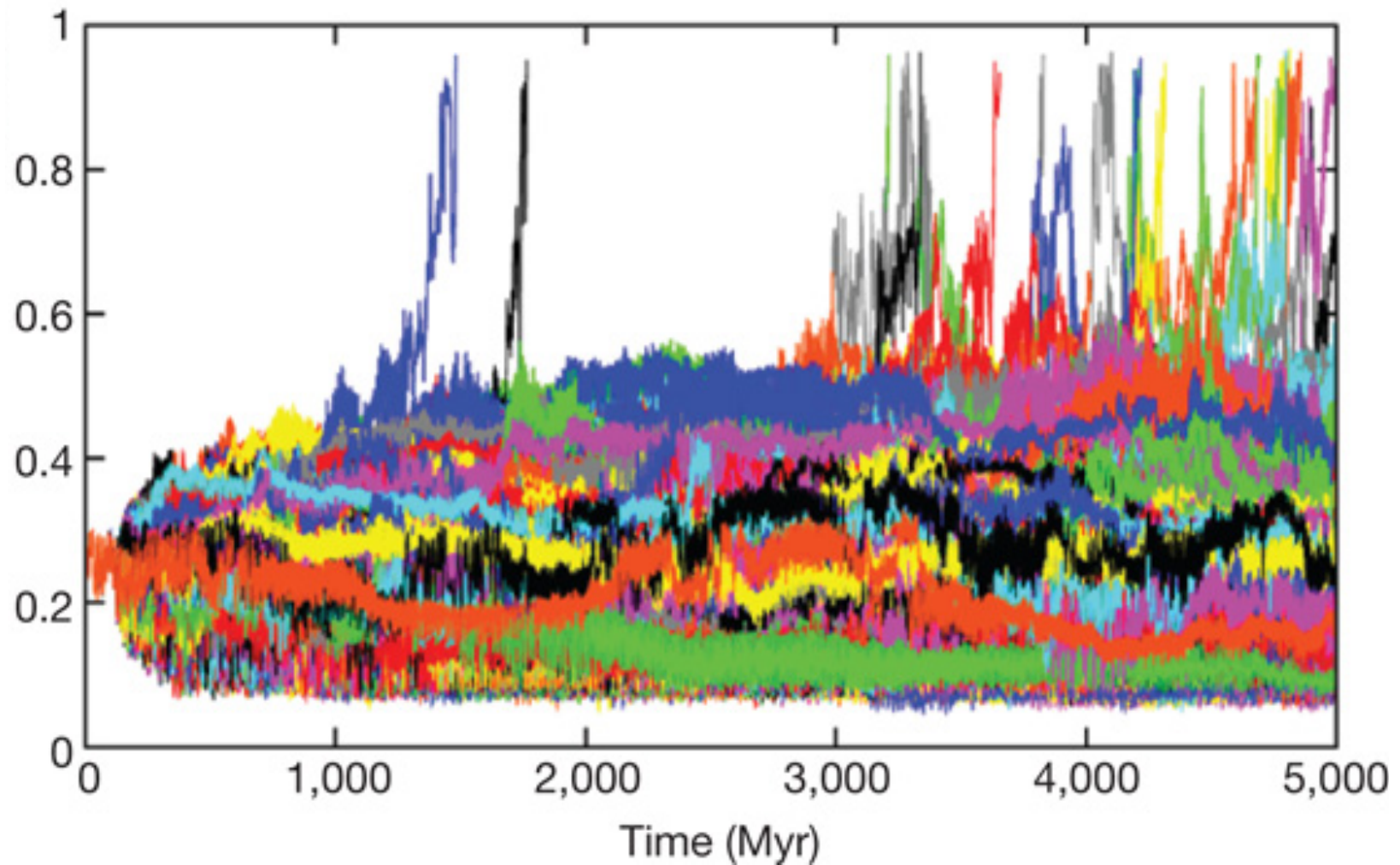
The workhorse for long orbit integrations in planetary systems is the **mixed-variable symplectic (MVS) integrator** (Wisdom & Holman 1991)

- what it does well: long (up to Gyr) integrations of planets on orbits that are not too far from circular and don't come too close
- what it doesn't do well: close encounters and highly eccentric orbits

The most popular public software packages for solar-system and other planetary integrations are MERCURY (John Chambers) and SWIFT (Hal Levison, Martin Duncan)

- include several integrators: MVS, Bulirsch-Stoer, Forest-Ruth, etc.
- can handle close encounters + test particles
- can include most important relativistic corrections

Following 9 planets for  $10^6$  yr takes about 30 minutes



eccentricity of Mercury over 5 Gyr from 2,500 integrations  
differing by  $< 1$  mm in semi-major axis of Mercury

(Laskar & Gastineau 2009)



# Higher-order symplectic integrators

“drift” and “kick” operators

$$D_h : q \rightarrow q' = q + hp, \quad K_h : p \rightarrow p' = p - h\nabla V(q')$$

- modified Euler method  $K_h D_h$  or  $D_h K_h$  (first-order integrator)
- leapfrog  $D_{h/2} K_h D_{h/2}(z)$  (second-order integrator)
- Forest method  $D_a K_a D_b K_c D_b K_a D_a$  (fourth-order integrator)  
if  $a = 1.35120h, b = -0.3512h, c = -1.7024h$
- automatically symplectic since  $D_h, K_h$  are symplectic
- any symmetric formula is time-reversible
- only one set of phase-space coordinates has to be stored
- can be generalized to arbitrarily high order (Yoshida 1993)

## Leapfrog with variable timestep (1)

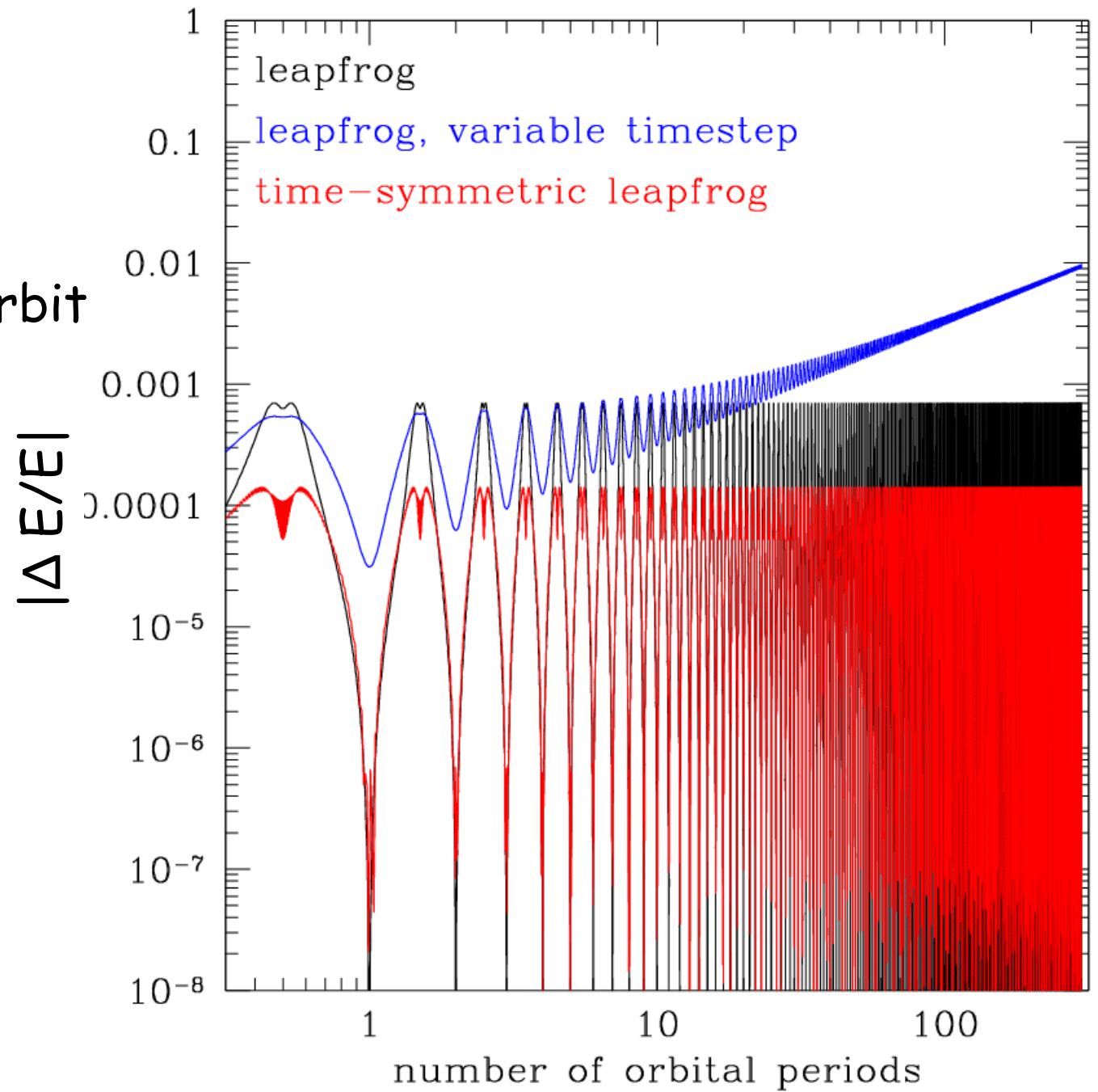
- we want to allow a variable timestep that depends on phase-space position,  $h = \tau(r, v)$
- time-reversible integrators have almost all the good properties of symplectic integrators
- define a symmetric function  $s(h, h')$ , e.g.  $s(h, h') = (h + h')/2$

$$\begin{aligned} \mathbf{r}' &= \mathbf{r}_n + \frac{1}{2}h\mathbf{v}_n & ; & & \mathbf{v}' &= \mathbf{v}_n + \frac{1}{2}h\mathbf{F}(\mathbf{r}') \\ s(h, h') &= \tau(\mathbf{r}', \mathbf{v}') \\ \mathbf{v}_{n+1} &= \mathbf{v}' + \frac{1}{2}h'\mathbf{F}(\mathbf{r}') & ; & & \mathbf{r}_{n+1} &= \mathbf{r}' + \frac{1}{2}h'\mathbf{v}_{n+1} \end{aligned}$$

This is time-reversible but not symplectic

$e=0.5$

200 steps per orbit



## Leapfrog with variable timestep (2)

### Time transformation:

- we want to allow a variable timestep that depends on phase-space position  $h = \tau(q,p)$
- introduce a new time variable  $t'$  by  $dt = \tau(q,p) dt'$ ; then unit timestep in  $t'$  corresponds to desired timestep in  $t$
- introduce extended phase space  $Q = (q_0, q)$  with  $q_0 = t$  and  $P = (p_0, p)$  with  $p_0 = -H$ . Then set

$$H'(Q,P) = \tau(q,p)[H(q,p) + p_0]$$

If  $(q,p)$  satisfy Hamilton's equations with Hamiltonian  $H$  and time  $t$ , then  $(Q,P)$  satisfy Hamilton's equations with Hamiltonian  $H'$  and time  $t'$

- works very well on eccentric orbits but only for one particle (can't synchronize timesteps of different particles)

## Leapfrog with variable timestep (3)

- we have a general differential equation  $dy/dt = f(t,y)$  that is known to be time-reversible
- we want an integration scheme that is time-symmetric with a variable timestep that depends on  $y$ ,  $h = \tau(y)$
- define a symmetric function  $s(h,h')$ , e.g.  $s(h,h') = (h+h')/2$
- pick your favorite one-step integrator,  $y_{n+1} = y_n + g(y_n, h)$  (e.g. Runge-Kutta)
- introduce a dummy variable  $z$  and set  $z_n = y_n$  at step  $n$

$$\begin{aligned} y' &= y_n + g(z_n, h/2) & ; & & z' &= z_n - g(y', -h/2) \\ s(h, h') &= \tau(y') \\ z_{n+1} &= z' + g(y', h/2) & ; & & y_{n+1} &= y' - g(z_{n+1}, -h/2) \end{aligned}$$

This is time-reversible (Mikkola & Merritt 2006)



# Summary

When integrating ordinary differential equations

- short-term **quantitative** accuracy is not the same as---and is often less important than---long-term **qualitative** accuracy
- use geometric integrators, which preserve the qualitative features of the physical systems they are describing (symplecticity, time-reversibility, etc.)
- if the physical system is close to one that can be integrated exactly, choose the integration algorithm so that it is exact for the integrable system
- implement variable timestep in a time-reversible algorithm