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Some comments on the recent twistor space constructions

In this talk I would like to comment on certain features of the Horrocks-Barth-Atiyah-Hitchin-Manin-Drinfeld (HBAHMD) construction [1] of bundles in $\mathbb{CP}^3$ and self-dual Yang-Mills solutions. (This construction will also be described at this conference by Robin Hartshorne.)

One purpose is to explain in a simple language why this construction works, and incidentally to translate the twistor space construction into Minkowski space language. However, the main motivation for the effort described here is that I feel that the construction as usually presented has a number of features that are too special for broad applicability in physics.

In particular, I would like to de-emphasize those aspects of the construction that are inherently global. Although the treatment of the global Euclidean space problem is very powerful and beautiful, we must remember that physics takes place in Minkowski space and that global considerations are not usually in the forefront in physics. Also, the quantum Yang-Mills field satisfies not the self-dual equation $F_{\mu\nu} = \tilde{F}_{\mu\nu}$ but the second order equation $D^{\mu}F_{\mu\nu} = 0$. If we hope to eventually make applications to quantum field theory, we must probably learn to think about this latter equation in Minkowski space. And this means not a global problem but a local or initial value problem. In particular, it would be very exciting to find an analogue of the HBAHMD construction for the Minkowski space equation $D^{\mu}F_{\mu\nu} = 0$.

With this in mind, I will be describing the HBAHMD construction in a way that is suited for a consideration of the local problem--that is, the problem of finding self-dual Yang-Mills fields that are well defined, not throughout Euclidean space, but just in a small open set thereof. (This corresponds, in $\mathbb{CP}^3$, to a bundle defined not on all of $\mathbb{CP}^3$ but just on a neighborhood of a line.)

First, let us review Ward's construction of the self-dual Yang-Mills solutions [2]. Ward considers (cf. his paper in this volume) the space of all left-handed null two planes α in four dimensional complex Minkowski space $\mathbb{CM}$. Given a self-dual gauge field in $\mathbb{CM}$, Ward introduces a vector bundle on $\mathbb{CP}^3$ as follows. An element of this bundle is a pair (α,ϕ) where α is a null two plane in $\mathbb{CM}$ and ϕ is a (Lorentz) scalar field (in the fundamental representation of the gauge group) that is covariantly constant on α.

We may say that ϕ is defined only on α. But a statement that is more suitable for generalizations is to say that ϕ is defined throughout $\mathbb{CM}$ (rather, throughout as much of $\mathbb{CM}$ as we are working with) but is defined only modulo the addition of a field vanishing on α. According to this definition, in other words, the fiber of Ward's bundle over a given two plane α is the space of all scalar fields ϕ with $D_\mu\phi = 0$ on α, modulo the space of all fields that vanish on α. The bundle E consisting of all pairs (α,ϕ) has, as Ward showed, a natural holomorphic structure, from which the original self-dual Yang-Mills gauge field can be reconstructed.

Now, HBAHMD tell us that this bundle E can be obtained as follows. We introduce vector spaces A, B, and C, and for each point Z^α in $\mathbb{CP}^3$ we introduce linear maps $f(Z):A \to B$, $g(Z):B \to C$. For fixed Z, f and g are linear maps, and also they depend linearly on the homogeneous coordinates Z^α of $\mathbb{CP}^3$. Also, for fixed Z, $g(Z)f(Z) = 0$. The whole picture is

$$A \xrightarrow{f(Z)} B \xrightarrow{g(Z)} C \tag{1}$$

and the bundle E arises as the kernel of g modulo the image of f. In other words, for fixed Z, the fiber of E at Z is the kernel of g modulo the image of f.

I would like to explain what A,B,C,f, and g are in the construction of reference [1], and why they are that way.

As a first approximation, let us imagine that A, B, and C each are the space of all scalar fields (in the fundamental representation of the gauge group) and let us try to define maps f and g such that E will be the kernel of g divided by the image of f.

We will define f, corresponding to a given Z or to a given two plane α, to be multiplication by a suitably chosen function which vanishes on α. Then im f will consist precisely of scalar fields that vanish on α.

It is a little harder to define g. We will define g so that the kernel of g consists of all scalar fields that are covariantly constant on α. Then ker g/im f will be exactly the space of all covariantly constant fields on α.

The above is a good approximation to what we want, but it suffers from a basic deficiency. We are trying to construct self-dual gauge fields. A self-dual gauge field is completely determined by certain arbitrary functions of three variables, which one can choose, for instance, to be the values of the gauge field on the initial value hypersurface $t = 0$. However, the spaces A, B, and C, as defined above, are spaces of arbitrary functions of four variables. At best, it is extremely redundant to describe the self-dual gauge fields, which really depend only on functions of three variables, in terms of spaces A, B, and C that involve arbitrary functions of four variables.

The problem is not just a problem of redundancy; it is a problem of principle. If our goal is to find a construction along the lines of (1) that could be used, at least in principle, to solve the initial value problem, then A, B, C, f, and g should be spaces, and maps, that are known explicitly once the initial data are given. I think that this is a reasonable property to insist on in a construction like [1].

To arrange that A, B, and C are spaces of functions of three variables, not four, and that they (and the maps f and g between them) are such as to be known explicitly once the initial data are given, we should choose A, B, and C to be spaces, not of arbitrary scalar functions, but of solutions of some auxiliary equations in the background self-dual gauge field. Then A, B, and C will be spaces of certain arbitrary functions of three variables--the initial data of the auxiliary equations.

To proceed further, I must be more specific, and tell you what function that vanishes on α we will use in defining f.

The function we use will be not a scalar but a spinor,

$$\psi_A = c_A + x_{AA'} d^{A'} \tag{2}$$

where $(c_A, d^{A'})$ are constants, the choice of which depends on α, and where $x_{AA'}$ are the coordinates of complex Minkowski space.

Now, you will recognize that $\psi_A = 0$ defines a two dimensional surface which is in fact a left-handed null two plane--a surface corresponding to a twistor or element of $\mathbb{CP}^3$. (In fact, c_A and $d^{A'}$ can be regarded as the homogeneous coordinates of $\mathbb{CP}^3$.)

We will sometimes identify a two plane α with the spinor (2) that vanishes on it.

I will also need the spinors of opposite helicity

$$\tilde{\psi}_{A'} = \tilde{d}_{A'} + x_{AA'} \tilde{c}^{A} . \tag{3}$$

It is an essential, and not completely obvious, fact that the spinor spaces ψ and $\tilde{\psi}$ are dual to each other. In fact, we can define a bilinear form

$$(\psi,\tilde{\psi}) = \tilde{d}_{A'}d^{A'} - c_A\tilde{c}^A \quad . \tag{4}$$

This form is obviously Lorentz invariant. It is not obvious that it is conformally invariant, but this is, in fact, true.

Now, let us define the spaces A, B, and C.

A will be a space of unprimed spinors u_A that satisfy a certain equation that will be described later.

B will be a space of scalar fields ϕ that satisfy a certain equation that also will be described later.

C will be the space of the unprimed spinors w_A which satisfy the Dirac equation

$$D^A{}_{A'}w_A = 0. \tag{5}$$

Now let us define our maps f and g. Remember, f and g will depend on the choice of two plane in $\mathbb{C}\mathbb{M}$ or, equivalently, on the choice of the spinor ψ_A which vanishes on the two plane.

We define $f:A \to B$ by

$$f(\psi)u_A = \psi^A u_A \tag{6}$$

Then the image of f consists of functions that vanish on the two plane $\psi^A = 0$.

To define g I must first state what equation the scalar fields ϕ in B satisfy. This equation will be given in a rather implicit form (but see eq. (13) for a more explicit version) and at first sight looks rather peculiar.

The condition to be satisfied by ϕ is that its covariant derivative has an expansion

$$D_{AA'}\phi = \sum_{a=1}^{4} \tilde{\psi}_{A'}{}^{a} w_A{}^{a} \tag{7}$$

where $\tilde{\psi}_{A'}{}^{a}$, a=1, ... ,4 are our four spinors of primed type defined in equation (3), and where $w_A{}^{a}$, a=1, ... ,4 are required to be solutions of the Dirac equation, that is, elements of C.

Now we can define our map $g(\psi):B \to C$. We define

$$g(\psi)\phi = \sum_{a=1}^{4} (\psi,\tilde{\psi}^{a}) w_A{}^{a}$$

We must show that the kernel of g consists precisely of those scalar functions that are covariantly constant on the two surface on which ψ vanishes.

To argue this, I will proceed in a noncovariant way. Consider the special case $c_A = 0$, $d^{1'} = 1$, $d^{2'} = 0$, that is $\psi_A = x_{A1'}$. Any case could be mapped onto this one by a Lorentz transformation.

Also, I will choose a basis for the primed spinors $\tilde{\psi}_{A'}$:

$$\begin{aligned} &\tilde{\psi}^{(1)}_{1'} = 1, \qquad \tilde{\psi}^{(1)}_{2'} = 0 \\ &\tilde{\psi}^{(2)}_{1'} = 0, \qquad \tilde{\psi}^{(2)}_{2'} = 1 \\ &\tilde{\psi}^{(3)}_{A'} = x_{1A'} \\ &\tilde{\psi}^{(4)}_{A'} = x_{2A'}. \end{aligned} \tag{9}$$

In this basis the formula $D_{AA'}\phi = \sum_a \tilde{\psi}_{A'}{}^{a} w_A{}^{a}$ means that

$$D_{A1'}\phi = w_A{}^{(1)} + x_{11'}\, w_A{}^{(3)} + x_{21'}\, w_A{}^{(4)}. \tag{10}$$

Also, $(\psi,\tilde{\psi})$ is nonzero, with our choice of ψ and in this basis, only for $\tilde{\psi}^{(1)}$. Therefore $g(\psi)\phi$, with out definitions, is zero if and only if $w_A^{(1)} = 0$. In this case

$$D_{A1'}\phi = x_{11'} w_A^{(3)} + x_{21'} w_A^{(4)} \tag{11}$$

so that

$$D_{A1'}\phi = 0 \text{ if } x_{B1'} = 0. \tag{12}$$

This is the desired result, stating that to be in the kernel of $g(\psi)$, ϕ must be covariantly constant on the two plane on which ψ vanishes.

As one can see by differentiating equation (7) and symmetrizing on the primed indices, antisymmetrizing on the unprimed ones (and using the fact that w_A satisfies the Dirac equation while $\psi_{A'}$ satisfies the twistor equation $\partial_{AA'}\tilde{\psi}_{B'} + \partial_{AB'}\tilde{\psi}_{A'} = 0$) to get a consistency condition, fields ϕ that satisfy (7) exist only if the background gauge field with respect to which D_μ is defined is self-dual; it is here that self-duality enters into the construction.

Now let us tie up the loose ends in the definitions of A, B, and C. C was already defined as the space of solutions of the Dirac equation (5). B we define as the space of all scalar fields whose covariant derivatives can be expanded in the form (7), and A we define as the space of spinors u_A such that, for any ψ^A of type (2), $u_A\psi^A$ is an element of B. These definitions are rather indirect and are not immediately transparent. With some thought one can see that a more explicit and equivalent way to define B is to say that B consists of the space of all scalar fields ϕ that can be written

$$\phi = a + x^{AA'} b_{AA'} + x^2 c \tag{13}$$

where a, $b_{AA'}$, and c all satisfy the covariant Laplace equation, $D_\mu D^\mu s = 0$. However, a given element of B can be written in the form (13) in many ways, as a result of which this expansion is awkward to use. The general element of A can be defined in a somewhat similar way.

Since we have shown that the kernel of g consists of scalar functions covariantly constant on α, and the image of f consists of scalar functions vanishing on α, it may seem quite plausible that the kernel of g modulo the image of f is precisely the fiber of Ward's bundle E. What still must be shown is that B is large enough that any desired covariantly constant value on α is assumed by some element of B, and that A is large enough relative to B that the equivalence classes, the kernel of g modulo the image of f, depend only on the value on α. These facts can be established by relatively elementary arguments analogous to those above.

Rather than continuing in this vein, let us shift here to a more formal line of argument. Manin and Drinfeld [1] have given a very elegant derivation of the construction under discussion here using the Kazul complex, which is an exact sequence of sheaves. Let Ω^k be the k^{th} antisymmetric tensor product of the cotangent bundle of $\mathbb{CP}^3$. Then there is an exact sequence of sheaves

$$0 \to \Omega^3(3) \to \Omega^2(2) \to \Omega^1(1) \to \mathbb{C} \to \mathbb{C}|_S \to 0 \tag{14}$$

depending on the arbitrary choice of a point S in $\mathbb{CP}^3$ (and where $\mathbb{C}|_S$ represents the complex numbers sitting at S). Following Manin and Drinfeld, one tensors (14) with E(-1) and writes down the standard long exact sequences of cohomology groups, in order to obtain information about E. In this way one finds that if one defines

$$A = H^1(E \otimes \Omega^2(1))$$

$$B = H^1(E \otimes \Omega) \qquad (15)$$

$$C = H^1(E(-1))$$

and f and g as the natural maps induced from (14), then E is in fact ker g/im f.

The point I wish to make here is that one can easily see that global properties are not needed in this argument. If instead of all of $\mathbb{CP}^3$ one is working with a neighborhood of a line in $\mathbb{CP}^3$, the Manin-Drinfeld argument still goes through. (Some of the arguments for why certain cohomology groups vanish are modified, but the net conclusion is the same.) A neighborhood of a line in $\mathbb{CP}^3$ corresponds to a small open set in complex Minkowski space, and thus it corresponds to the problem of a self-dual solution that is defined only in a small open set. Thus, the Manin-Drinfeld derivation shows that the construction that we are discussing here is valid for this local problem as well as for the global problem.

In fact, from the Minkowski space point of view suggested in this talk, the construction is "obviously" right in the local problem, while its validity in the global case is a dèlicate fact. The strategy described in the first three paragraphs after equation (1), and followed in this talk, will certainly work if A, B, and C are defined to be large enough spaces of functions. But if they are too small it could happen that ker g/im f would be not the whole fiber of E, but only a subspace. It is most clear that this does not happen if A, B, and C are defined without restriction as spaces of functions of four variables, as suggested in the comments just after equation (1). With the definitions actually given in this paper, A, B, and C being defined in terms of functions of three variables, it is slightly delicate but still relatively easy to see that the construction works. In the global problem treated in reference one, where the elements of A, B, and C are required not only to satisfy conditions (5), (6), and (7), but also to be global elements of the cohomology classes, these spaces become finite dimensional. In this case it is a quite delicate fact that the spaces A, B, and C are still large enough that the construction works.

Thus, this construction is also valid in a local form. As I mentioned at the outset of this talk, the reason that I think that this fact has some significance is related to the fact that the second order Minkowski space Yang-Mills equation, $D^{\mu}F_{\mu\nu} = 0$, is quite probably the equation we must come to grips with if we hope for applications to quantum field theory. These equations are hyperbolic; one can think of them in terms of a local problem or an initial value problem, but there is for them no global problem analogous to the global self-dual problem on S^4. The fact that the HBAHMD construction makes sense in a local version encourages one to hope that an analogue of this construction may exist for the second order Yang-Mills equations. The discovery of such an analogue would be very exciting.

A suitable starting point in a search for such a construction might be a construction recently found for the second order Yang-Mills equations by Yasskin, Isenberg, and Green [3] and by Witten [4]. This construction has been presented at this conference in talks by Yasskin and by Isenberg.

I would like to conclude with several technical remarks. In the local problem, A, B, and C are infinite dimensional vector spaces. The use of infinite dimensional spaces should not alarm or surprise us. It is analogous to the fact that in the inverse scattering appraoch to the sine-gordon equation, the sine-gordon field is reconstructed form linear maps among certain infinite dimensional vector spaces, the spaces of solutions of the auxiliary Schroedinger equation [5].

It should be noted that in the local problem, f is not injective, although g is still surjective. If one wants to find a sequence of spaces and maps exact except at one stage, the non-exactness corresponding to E, then it is necessary to introduce a fourth vector space; roughly speaking, the fourth space is the kernel of f. Also, it should be noted that certain simplifying features of this construction that appear in the global case are missing in the local case: A is not dual to C, B does not have a natural bilinear form, and g is not the adjoint of f (in the global problem these properties appear if E is symplectic or orthogonal).

The last point to be made here concerns the question of the choice of gauge groups or of the rank of vector bundles. Most discussion so far has concerned SU(2) gauge fields or rank two bundles. For applications to quantum field theory, however, the often made assumption that SU(2) is the simplest theory is probably an error.

Rather, it was shown by 't Hooft in a very important paper [6] that has not received all the attention it deserves that Yang-Mills theory with an SU(N) gauge group is, as a quantum field theory, simplest in the limit $N \to \infty$. I believe that the most realistic goal in this field is to try to understand the large N limit proposed by 't Hooft.

It is very striking that the HBAHMD construction also simplifies (for fixed value of the Chern class) as $N \to \infty$ [7]. This convergence of the region in which the quantum field theory is known to simplify and the region in which the mathematics of classical solutions simplifies is very intriguing, and perhaps of great significance. It may well turn out to be, in the long run, the most significant aspect of the construction, if one has physical applications in mind.

I would conjecture that a generalization of the HBAHMD construction to the second order equations, if it exists, will be something that is uselessly awkward for any finite N, and is tractable only at N = ∞.

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Acknowledgment

I would like to thank D. Kazhdan and D. Mumford for discussions.

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