HOLOMORPHIC MORSE INEQUALITIES*

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Given a holomorphic vector field V on a compact complex manifold M, the Atiyah-Bott holomorphic Lefschetz formula expresses the Chern numbers of M in terms of the zeros of V. In this article, it is shown that if M is a Kahler manifold and V generates an isometry of M, the holomorphic Lefschetz formula can be generalized to a system of inequalities, analogous to the Morse inequalities for real manifolds.

The celebrated Morse inequalities bound the cohomology of a manifold in terms of the critical points of an arbitrary smooth function h. Thus, let N be a compact oriented manifold of dimension n. Let B_k be the k^{th} Betti number of N. Let M_k be the k^{th} Morse number of the function h — that is, the number of critical points that are unstable in k directions. Let t be an arbitrary real number. Then the Morse inequalities assert that

$$\sum_{k=0}^{n} M_k t^k = \sum_{k=0}^{n} B_k t^k + (1+t) \sum_{k=0}^{n-1} Q_k t^k \tag{1}$$

where the Q_k are non-negative integers that depend on h.

We could instead consider the Morse function −h. This changes the sign of the Hessian at each critical point, so it exchanges M_k with M_{n-k}. The result gained by applying (1) to the Morse function −h is

$$\sum_{k=0}^{n} M_{n-k} t^k = \sum_{k=0}^{n} B_k t^k + (1+t) \sum_{k=0}^{n-1} \tilde{Q}_k t^k \tag{2}$$

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and this is precisely what we could have learned by supplementing (1) by Poincaré duality ($B_k = B_{n-k}$) and replacing t with 1/t.

In the special case t = −1, the Morse inequalities give a formula

$$\sum_{k=0}^{n} (-1)^k M_k = \sum_{k=0}^{n} (-1)^k B_k \qquad (3)$$

for the Euler characteristic of the manifold M. This formula is essentially a special case of Hopf's theorem expressing the Euler characteristic of N in terms of the zeros of an arbitrary vector field. (Introducing an arbitrary Riemannian metric γ on M, the zeros of the "gradient" vector field $W^i = \gamma^{ij} \frac{\partial h}{\partial \phi^j}$ are the critical points of h; and for a critical point of Morse index k, the corresponding zero of W has index $(-1)^k$.)

Hopf's theorem has a generalization to complex manifolds — the Atiyah-Bott fixed point theorem[1] or holomorphic Lefschetz formula. Let M be a compact complex manifold of complex dimension n, and let V be a holomorphic vector field. Let E be a holomorphic vector bundle over M, such that the action of V on M can be lifted to a holomorphic action on E. Let H^k be the k^{th} $\bar\partial$ cohomology group of M, with coefficients in E.

In this situation, the action of V on E induces a linear transformation on H^k. Let us denote this linear transformation as S_k and define $H^k(\theta) = \mathrm{Tr}_{H^k} \exp(i\theta S_k)$.

Although the holomorphic Lefschetz formula does not require these assumptions, let us, with later applications in mind, specialize to the case that M is a Kahler manifold and V leaves invariant some Kahler metric on M. V is then a generator of a compact isometry group. For simplicity only (this assumption will play no crucial role in what follows) let us assume that V generates a U(1) action on M. Let us assume as well (the relaxing of this assumption is discussed at the end of this paper) that the zeros of V are isolated points $p^1 \ldots p^q$. Then near any zero p^a, we have $V = \sum_{i=1}^{n} \lambda_i^a z^i \frac{\partial}{\partial z^i}$ with some non-zero integers λ_i^a and local holomorphic coordinates z^i.

The action of V on E induces a linear transformation, which we will call T_a, on the fiber $E(p^a)$ of E at p^a. Let $E_a(\theta) = \mathrm{Tr}_{E(p^a)} \exp(i\theta T_a)$. And finally, define the "index" n^a of the vector field V at p^a to be the number of λ_i^a that are negative.

We can finally state the holomorphic Lefschetz formula:

$$\sum_a (-1)^{n_a} E_a(\theta) \prod_{\lambda_i^a>0} \frac{1}{1 - e^{i\lambda_i^a\theta}} \prod_{\lambda_i^a<0} \frac{e^{-i\lambda_i^a\theta}}{1 - e^{-i\lambda_i^a\theta}} = \sum_k (-1)^k H^k(\theta) . \tag{4}$$

On the left-hand side of (4), the "holomorphic determinant" has been written in a slightly unusual way that will be useful.

Now equation(4) is a sort of generalization to complex manifolds of Hopf's formula for the Euler characteristic of a real manifold. And Hopf's formula appears in Morse theory by setting t = -1 in the Morse inequalities. Our goal in this article is to establish a system of inequalities which one might think of as a holomorphic form of the Morse inequalities; the holomorphic Lefschetz formula of equation (4) follows from the holomorphic Morse inequalities by setting t = -1, just as (a special case of)Hopf's formula follows from the usual Morse inequalities in this way. The inequalities that we will establish require one assumption not yet stated --we must suppose V does have at least one zero.

The holomorphic Morse inequalities which will be established here assert that

$$\sum_a t^{n_a} E_a(\theta) \prod_{\lambda_i^a>0} \frac{1}{1 - e^{i\lambda_i^a\theta}} \prod_{\lambda_i^a<0} \frac{e^{i|\lambda_i^a|\theta}}{1 - e^{i|\lambda_i^a|\theta}}$$

$$= \sum_k t^k H^k(\theta) + (1+t) \sum_k t^k Q_k(\theta) \tag{5}$$

where each function $Q_k(\theta)$ is positive in the sense that Q_k has a Fourier expansion

$$Q_k(\theta) = \sum_{m=-\infty}^{\infty} Q_{km} e^{im\theta} \tag{6}$$

in which each coefficient Q_{km} is a non-negative integer. ($Q_{km}(\theta)$ will in general be singular at $\theta = 0$. As the discussion will make clear, the resulting ambiguity in the Fourier expansion of $Q_k(\theta)$ is to be settled by analytically continuing in θ and writing an expansion that converges for $0 < |z| < 1$ where $z = e^{i\theta}$.)

Setting $t = -1$, the inequalities (5) reduce to the holomorphic Lefschetz formula of equation (4). But there is one important difference between the usual Morse inequalities and the holomorphic ones. While in many important cases, the usual Morse inequalities are exact and the Q_k of equation (1) vanish, this never occurs in the holomorphic case. On the contrary, there are always an infinite number of non-zero Q_{km} in equation (6).

To obtain strong information in the holomorphic case, we must supplement equation (5) with additional information that can be obtained by considering the vector field $-V$ instead of V. Replacing V by $-V$, all of the λ_i^a change sign; if one also substitutes $\theta \to -\theta$ the inequality (5) for the vector field $-V$ is

$$\sum_a t^{n-n_a} E_a(\theta) \prod_{\lambda_i^a>0} \frac{e^{-i\lambda_i^a\theta}}{1 - e^{-i\lambda_i^a\theta}} \prod_{\lambda_i^a<0} \frac{1}{1 - e^{-i|\lambda_i^a|\theta}}$$

$$= \sum_k t^k H_k(\theta) + (1+t) \sum_k t^k \tilde{Q}_k(\theta) \qquad (7)$$

where $\tilde{Q}_k$ is again positive in the sense of equation (6) (one takes an expansion of $\tilde{Q}_k$ that converges for $|e^{i\theta}|>1$). The relation between (7) and (5) is like the relation between (2) and (1). (7) differs from (5) by Serre duality in the following sense. Let $\tilde{E}$ be the dual bundle of E tensored with the canonical line bundle of M. Replacing E by $\tilde{E}$ in (5), using Serre duality, and substituting $t \to 1/t$, one arrives at (7).

As we will see, (5) and (7) taken together are very restrictive; we will give some examples of "perfect vector fields" — examples in which (5) and (7) are strong enough to determine the $H_k(\theta)$ in terms of $E_a(\theta)$ and the λ_i^a.

Now let K be the Kahler form of the Kahler manifold M. It is a closed form of type (1,1). Let $\mu = i_V \cdot K$ be the interior product of the holomorphic vector field V with K. μ is a form of type (0,1). Then $\bar\partial\mu = 0$ (since V is holomorphic and K is closed), and therefore at least locally we can write $\mu = \bar\partial\phi$ with some zero-form ϕ.

In this article we will restrict ourselves to the case in which $\mu = 0$ as an element of $H^{0,1}(M)$ so that $\mu = \bar\partial\phi$ can be satisfied globally. This is certainly true if the first Betti number of M is zero, for then $H^{0,1}(M) = 0$. It is also true if V has at least one zero, for in this case the orbits generated by the action of V on M (which are closed, since we assume V generates a U(1) action) vanish in $\pi^1(M)$, and μ vanishes in H^1 and in $H^{0,1}$.

From the assumption that V generates an isometry of M, it follows that ϕ can be chosen to be real. For, if μ^* is the form of type (1,0) that is complex conjugate to μ, the condition that V is an isometry is $\partial\mu + \bar\partial\mu^* = 0$. In terms of ϕ this means $\partial\bar\partial\phi + \bar\partial\partial\phi^* = 0$, so the imaginary part of ϕ obeys $\partial\bar\partial \, \mathrm{Im}\phi = \bar\partial\partial \, \mathrm{Im}\,\phi = 0$. For compact M this means that $\mathrm{Im}\phi$ is a constant which we can choose to be zero.

Let us think of the smooth, real-valued function ϕ as a Morse function on M. Its critical points are precisely the zeros of V. ϕ is a non-degenerate Morse function if and only if the zeros of V are isolated, and in this case it is easy to see that ϕ is necessarily a perfect Morse function. For, near an isolated zero p^a where V can be written $V = \Sigma\, \lambda^i z^i \frac{\partial}{\partial z^i}$ we have $\phi = \Sigma\, \lambda^i |z^i|^2$. The Morse index of ϕ at p^a is twice the number of negative λ, since each z^i corresponds to two real coordinates. Since the Morse index of ϕ is even at each critical point, it follows, by the lacunary principle of Morse theory, that ϕ is a perfect Morse function — its Morse indices give the Betti numbers exactly.

What we will now see is that ϕ also plays a role as a "holomorphic Morse function." As indicated earlier, we will discuss the $\bar\partial$ cohomology of M with coefficients in an arbitrary holomorphic bundle E that admits the action of V. Let E^q be the tensor product of E with the bundle of forms of type (o,q). We have the $\bar\partial$ operators

$$\bar\partial : E^q \to E^{q+1} \tag{8}$$

and the cohomology group H^q defined as a quotient

$$H^q = \frac{\ker\left(\bar\partial : E^q \to E^{q+1}\right)}{\operatorname{im}\left(\bar\partial : E^{q-1} \to E^q\right)} \tag{9}$$

Now, by analogy with a recent discussion of the Morse inequalities,[2] introduce an arbitrary real number s and define $\bar\partial_s : E^q \to E^{q+1}$ by

$$\bar\partial_s = (\exp - s\phi)\ \bar\partial\ (\exp s\phi) \tag{10}$$

We also define the quotient space

$$H^q{}_s = \frac{\ker\left(\bar\partial_s : E^q \to E^{q+1}\right)}{\operatorname{im}\left(\bar\partial_s : E^{q-1} \to E^q\right)} \tag{11}$$

Since $\bar\partial_s$ differs from $\bar\partial$ only by conjugation by the invertible operator $\exp s\phi$, the information contained in $H^q{}_s$ is independent of s in the sense that

$$H^q{}_s(\theta) = \operatorname{Tr}_{H^q{}_s}\ e^{i\theta V}, \tag{12}$$

is independent of s.

On the other hand, let us define

$$\bar\partial_s{}^* = (\exp s\phi)\ \bar\partial^*\ (\exp -s\phi)$$

$$\Delta_s = \bar\partial_s{}^*\,\bar\partial_s + \bar\partial_s\,\bar\partial_s{}^* \tag{13}$$

(The adjoint $\bar\partial^*$ is defined relative to a V-invariant Hermitian structure on E.) If we let $Y^q{}_s = \ker\left(\Delta_s : E^q \to E^q\right)$ then it follows from standard arguments (just as at s = 0) that $H^q{}_s$ can be identified with $Y^q{}_s$. It follows, then, that

$$H^q(\theta) = \operatorname{Tr}_{Y^q{}_s}\ e^{i\theta V} \tag{14}$$

for any s.

Just as in reference (2), the utility of this formulation is that for large s, the spectrum of Δ_s simplifies considerably; it can be calculated explicitly as an asymptotic series in powers of 1/s. By studying the large s behavior of Δ_s, we will obtain the inequalities stated earlier (equations (5) and (7)).

It is useful to work out a detailed formula for Δ_s. Let $\Delta = \bar\partial\bar\partial^* + \bar\partial^*\bar\partial$. Let $|V|^2$ denote the norm of the vector field V, relative to the Kahler metric. Let $\mathcal{L}_V$ be the infinitesimal generator of the action of V on E. (If E is constructed from the tangent or cotangent bundle of M, then $\mathcal{L}_V$ may be chosen to be the Lie derivative.) Then one easily calculates

$$\Delta_s = \Delta + s^2\,|V|^2 - s\,\mathcal{L}_V + s\,\Phi \tag{15}$$

where Φ is an operator of degree zero (no derivatives), the details of which depend on E and $\mathcal{L}_V$. By analogy with reference (2), the basic idea in our analysis will be that for $s \to \infty$, the term $s^2\,|V|^2$ in (15) forces the eigenfunctions of Δ_s with small eigenvalue to be concentrated near the zeros of V. The eigenvalue problem then simplifies and the spectrum can be obtained as an asymptotic expansion in powers of 1/s. Our goal is to bound the cohomology by counting the eigenvalues that vanish as $s \to \infty$.

However, the presence in (15) of a first order term ($s\,\mathcal{L}_V$) causes a subtlety that did not arise in reference (2). Δ_s actually has an <u>infinite</u> number of eigenvalues that converge (exponentially fast) to zero as $s \to \infty$. Moreover, the convergence to the large s behavior is nonuniform in the following sense. Although any given eigenfunction of Δ_s becomes concentrated near the zeros of V for large s, no matter how large s is, there are eigenfunctions with small eigenvalue that are not concentrated near the zeros of V.

To proceed, we must note that because V is assumed to generate an <u>isometry</u>, we have

$$[\mathcal{L}_V, \Delta_s] = 0\ . \tag{16}$$

We therefore may look for solutions of the joint eigenvalue problem

$$\begin{aligned} \mathcal{L}_V \psi &= n\psi \\ \Delta_s\, \psi &= \lambda\psi \end{aligned} \tag{17}$$

Note that the eigenvalues of $\mathcal{L}_V$ are integers, because we have assumed that $\mathcal{L}_V$ generates a circle action.

Our strategy will be to fix the integer n — the eigenvalue of $\mathcal{L}_V$ — and study the large s limit of Δ_s. For fixed n, $\mathcal{L}_V$ can be replaced by the integer n in (15), and the large s behavior of the resulting operator is straightforward — analogous to the problems treated in reference (2). The spectrum of Δ_s, for fixed n and large s, can be worked out by means of the Rayleigh-Schrodinger perturbation theory.[3] The low-lying eigenvalues are uniformly concentrated near the zeros of V, and all but a finite number of the eigenvalues diverge in proportion to s as s becomes large.

Let $E^{q,n}$ be the subspace of E^q consisting of form ψ such that $\mathcal{L}_V\psi = n\psi$. Let $a_{k,n}$ be the number of eigenvalues of Δ_s, restricted to $E^{q,n}$, which vanish in the limit of large s. We will determine $a_{k,n}$ shortly. Let $H^k(n)$ be the Fourier coefficients of the quantity $H^k(\theta)$ defined earlier. Thus,

$$H^k(\theta) = \sum_n e^{in\theta} H^k(n) \qquad (18)$$

The $H^k(n)$ are non-negative integers; $H^k(n)$ is simply the number of <u>zero</u> eigenvalues of Δ or Δ_s, restricted to $E^{k,n}$.

Since the number of eigenvalues that are identically zero for any s must be no bigger than the number of eigenvalues that vanish in the large s limit, we have immediately the inequality

$$a_{k,n} > H^k(n) \qquad (19)$$

This is analogous to the weak form of the Morse inequalities. But we can assert a stronger inequality. The eigenfunctions that correspond to eigenvalues that do not diverge when s is increased form a model for $\bar{\partial}$ cohomology (in other words, since $\bar{\partial}_s$ commutes with Δ_s, the $\bar{\partial}_s$ cohomology is unchanged if $\bar{\partial}$ is restricted to the low-lying eigenfunctions). Just as in conventional Morse theory, we can deduce from this an inequality much stronger than (19):

$$\sum_k a_{k,n}\, t^k - \sum_k H^k(n)\, t^k = (1+t) \sum_k Q_{k,n}\, t^k \qquad (20)$$

Here the $Q_{k,n}$ are non-negative integers.

Equation (20) is precisely the result announced earlier (equations (5) and (6)), provided that we can show that

$$\sum_{k,n} a_{k,n}\, t^k\, e^{in\theta} = \sum_a t^{n_a} E_a(\theta) \prod_{\lambda_i^a>0} \frac{1}{1-e^{i\lambda_i^a\theta}} \prod_{\lambda_i^a<0} \frac{e^{i|\lambda_i^a|\theta}}{1-e^{i|\lambda_i^a|\theta}} \tag{21}$$

(The series on the left-hand side converges for $|e^{i\theta}|<1$.) Our next goal is, therefore, to establish (21).

To understand how (21) comes about, assume first that E is a trivial line bundle, the complex manifold M is just the complex plane, and $V = \lambda z \frac{\partial}{\partial z}$. (In the following discussion, whenever E is trivial, we take $\mathcal{L}_V$ to be the Lie derivative on differential forms.) In this case, zero eigenvalues of Δ_s are either zero forms $F(z,z^*)$ with

$$\left(\frac{\partial}{\partial z^*} + s\lambda z\right) F(z,z^*) = 0 \tag{22}$$

or one forms $G(z,z^*)\, dz^*$ with

$$\left(-\frac{\partial}{\partial z} + s\lambda z^*\right) G(z,z^*) \tag{23}$$

The solutions of these equations are

$$F_m(z,z^*) = z^m \exp - (s\lambda zz^*)$$

$$G_m(z,z^*)\, dz^* = z^{*m}\, dz^* \exp\, (s\lambda zz^*) \tag{24}$$

with $m = 0,1,2,\ldots.$

Now let us take $s \to +\infty$. (The result of $s \to -\infty$ is discussed later.) In this case, if $\lambda > 0$ only the zero forms are normalizable. Since

$$\mathcal{L}_V\, (z^m \exp - \lambda szz^*) = m\lambda\, (z^m \exp - \lambda szz^*) \qquad m = 0,1,2,\ldots \tag{25}$$

we have $a_{o,n} = 1$ for $n = 0,\lambda,2\lambda,\ldots$ and zero otherwise, while $a_{1,n} = 0$ for all n. On the other hand, if $\lambda < 0$, only the one forms are normalizable, so $a_{o,n} = 0$ for all n. Since

$$\mathcal{L}_V\, (dz^*\, z^{*m} \exp - \lambda szz^*) = -\lambda(m+1)(dz^* z^{*m} \exp - \lambda szz^*) \tag{26}$$

we have in this case $a_{1,n} = 1$ for $n = -\lambda, -2\lambda, -3\lambda, \ldots$ and zero otherwise.

So we find

$$\lambda > 0 : \sum_{k,n} a_{k,n}\, t^k e^{in\theta} = \sum_{n=0}^{\infty} e^{in\lambda} = \frac{1}{1 - e^{i\lambda}}$$

$$\lambda < 0 : \sum_{k,n} a_{k,n}\, t^k e^{in\theta} = t \sum_{n=1}^{\infty} e^{-in\lambda} = \frac{t\, e^{in|\lambda|}}{1 - e^{in|\lambda|}} \tag{27}$$

for the case where V generates a rotation of the complex plane.

Now suppose that E is still a trivial line bundle, but that M is C^n with $V = \sum_{i=1}^{n} \lambda^i z^i \frac{\partial}{\partial z^i}$. The eigenvalue problem for Δ_s trivially separates into n one dimensional problems, and $\Sigma\, a_{k,n}\, t^k e^{in\theta}$ can be evaluated immediately — it is just the product of the one dimensional formulae (27). If r is the number of negative λ, we get

$$\sum_{k,n} a_{k,n}\, t^k e^{in\theta} = t^r \prod_{\lambda_i > 0} \frac{1}{1 - e^{i\lambda_i\theta}} \prod_{\lambda_j < 0} \frac{e^{i|\lambda_j|\theta}}{1 - e^{i|\lambda_j|\theta}} \tag{28}$$

Continuing to let E be a trivial line bundle, let us now choose M to be an arbitrary compact Kahler manifold of complex dimension n. We assume that V has isolated zeros p^a, near any one of which V can be approximated, in some holomorphic coordinates, as $V = \sum_{i=1}^{n} \lambda^{ia} z^i \frac{\partial}{\partial z^i}$. The eigenvalue problem for Δ_s now cannot be solved explicitly, but for large s the spectrum can be calculated in an asymptotic expansion in powers of 1/s. For large s, there are low-lying eigenfunctions near any zero p^a of V, and the leading approximation to the spectrum of Δ_s, on eigenfunctions concentrated near p^a, coincides with the spectrum of the operator on C^n that we discussed in the previous paragraph. Including the contribution of each zero p^a we therefore get

$$\sum_{k,n} a_{k,n}\, t^k e^{in\theta} = \sum_a t^{n_a} \prod_{\lambda_i^a > 0} \frac{1}{1 - e^{i\lambda_i^a\theta}} \prod_{\lambda_i^a < 0} \frac{e^{i|\lambda_i^a|\theta}}{1 - e^{i|\lambda_i^a|\theta}} \tag{29}$$

where n^a is the number of negative λ_i^a.

Finally, permitting E to be an arbitrary holomorphic bundle that admits a holomorphic action of V, the contribution of each zero p^a is just multiplied by the function $E_a(\theta)$ defined prior to equation (4). (Introducing a non-trivial E does not change the large s behavior of the spectrum near p^a, but it shifts the eigenvalues of $\mathcal{L}_V$, introducing a factor $E_a(\theta)$.) We therefore obtain the desired result (21), and this completes the demonstration of the holomorphic Morse inequalities (5).

As has been previously mentioned, there is a second inequality that can be obtained in this way. It is equation (7). Equation (7) can be obtained by applying (5) to the tensor product of the dual bundle of E with the canonical line bundle of M, and then using Serre duality. Alternatively, (7) follows if one takes $s \to -\infty$. (In this case, the unnormalizable forms in (24) become normalizable, and vice-versa, and the subsequent formulas are all modified, leading to equation (7).)

The remainder of this paper is devoted mainly to a few applications of these inequalities. The first application is aimed to give some feeling for the content of the holomorphic inequalities. We will give an example of what one might regard as the holomorphic analogue of a perfect Morse function.

Let M be CP^1, and let V be the generator of one of the rotations of CP^1. As we know, any such V has two zeros. Near one of them, which we shall call P, we have $V = + z \frac{\partial}{\partial z}$ with a suitable holomorphic coordinate z; near the other zero, Q, we have $V = - w \frac{\partial}{\partial w}$ with a suitable w.

Now, let L be a line bundle over CP^1 which admits the action of V. Of course we know what the possible line bundles are, but instead of using this knowledge, let us see what can be learned just by use of the holomorphic Morse inequalities.

Given that an action of V on L exists, an arbitrary additive constant enters in defining this action. We are free to define the action of V so that the action on the fiber of L at P is trivial, but then, for given L, the action on the fiber at Q is uniquely determined. We thus assume that $E_P(\theta) = 1$, while $E_Q(\theta) = e^{ir\theta}$, where r is an integer characteristic of L. We will see that the holomorphic Morse inequalities are strong enough to determine the $\bar{\partial}$ cohomology of CP^1 with coefficients in L in terms of r.

Let us first consider equation (5). The contribution of P to the left-hand side of (5) is $1/(1 - e^{i\theta})$ while the contribution of Q is $t\, e^{i(r+1)\theta}/(1 - e^{i\theta})$. So we get

$$\frac{1}{1 - e^{i\theta}} + \frac{t\, e^{i(r+1)\theta}}{1 - e^{i\theta}} = H^0(\theta) + t\, H^1(\theta) + (1+t)\, Q(\theta) \tag{30}$$

where $Q(\theta)$ has a Fourier expansion with non-negative coefficients. In more detail, (30) reads

$$\sum_{n=0}^{\infty} e^{in\theta} + t \sum_{n=r+1}^{\infty} e^{in\theta} = \sum_{n=-\infty}^{\infty} \left(H^0(n) + t\, H^1(n)\right) e^{in\theta}$$

$$+ (1+t) \sum_{n=-\infty}^{+\infty} Q^n\, e^{in\theta} \tag{31}$$

Here the Q^n, $H^0(n)$, and $H^1(n)$ are all non-negative integers.

To indicate the nature of the argument, let us first consider the special case $r > 0$. In this case we immediately see from (31) that $H^0(n) = H^1(n) = 0$ for $n < 0$, because of the positivity of Q^n and the fact that the coefficient of $e^{in\theta}$ vanishes in the left of (31) for $n < 0$.

For $n > 0$ (31) is too weak to determine the $H^k(n)$. To proceed, we must use inequality (7). Inequality (7) gives in this case

$$\frac{t\, e^{-i\theta}}{1 - e^{-i\theta}} + \frac{e^{ir\theta}}{1 - e^{-i\theta}} = H^0(\theta) + t\, H^1(\theta) + (1+t)\, \tilde{Q}(\theta) \tag{32}$$

where $\tilde{Q}$ has non-negative Fourier coefficients or (expanding the left-hand side in a series that converges for $|e^{i\theta}| > 1$)

$$\sum_{n=-\infty}^{-1} t\, e^{in\theta} + \sum_{n=-\infty}^{r} e^{in\theta} = \sum H^0(n)\, e^{in\theta} + t \sum H^1(n)\, e^{in\theta}$$

$$+ (1+t) \sum \tilde{Q}^n\, e^{in\theta} \tag{33}$$

We now see that the coefficient of $e^{in\theta}$ on the left-hand side vanishes for $n > r$, so $H^0(n) = H^1(n) = 0$ if $n > r$.

For $0 < n < r$ we see from either (31) or (33) that $H^1(n) = 0$ because for such values of n, the coefficient of $t\, e^{in\theta}$ vanishes on the left of both (31) and (33). We now know that $H^1(n)$ vanishes for all n, and we may invoke what is essentially the lacunary principle of Morse theory. Since, for 0 <

$n < r$, the coefficient of $e^{in\theta}$ on the left of (31) or (33) is independent of t, we must have $Q^n = \tilde{Q}^n = 0$ for $0 < n < r$. We can then read off from (31) or (33) that $H^0(n) = 1$ for $0 \leq n \leq r$.

We hence have

$$H^0(\theta) = \sum_{n=o}^{r} e^{in\theta} \quad ,$$

$$H^1(\theta) = 0 \tag{34}$$

for $r > 0$. The same sort of reasoning determines $H^0(\theta)$ and $H^1(\theta)$ for other values of r. For $r = -1$ we get $H^0(\theta) = H^1(\theta) = 0$, and for $r \leq -2$ we get

$$H^0(\theta) = 0$$

$$H^1(\theta) = \sum_{n=r+1}^{-1} e^{in\theta} \tag{35}$$

These conclusions are, of course, in agreement with known results. The bundle L is necessarily the r^{th} power of the Hopf bundle, and the above results agree with standard determinations of the cohomology of these line bundles.

As a further application, let us see that we can partly retrieve a theorem of Carrell and Liberman.[4] Those authors proved the following. Let M be a Kahler manifold, and suppose that on M there is a holomorphic vector field V with only isolated zeros. Then the cohomology groups $H^{k,\ell}$ are zero for $k \neq \ell$.

Their proof does not require the assumption that V generates an isometry of M. However, if we make this assumption, then the vanishing of $H^{k,\ell}$ for $k \neq \ell$ follows from the holomorphic Morse inequalities, as we will now see.

Let us first prove that $H^{0,k} = 0$ for $k \neq 0$. $H^{0,k}$ is simply the k^{th} $\bar{\partial}$ cohomology group of M with coefficients in the trivial line bundle. So to study $H^{0,k}$ we must set the functions $E_a(\theta)$ in equations (5) and (7) equal to 1.

In equation (5), we now see that any term proportional to a non-zero power of t has a positive power of $e^{i\theta}$. For the numerator of (5) contains an explicit factor of $e^{i|\lambda_i^a|\theta}$ for every negative λ_i^a, and at least one of them must be negative to give a positive power of t on the left side of (5). (Recall that the denominators in (5) are to be expanded in positive powers of $e^{i\theta}$.) This means that $H^{0,k}(\theta)$, for $k > 0$, contains only strictly positive powers of $e^{i\theta}$.

However, on the left-hand side of (7), any term proportional to a non-zero power of t is proportional to a strictly negative power of $e^{i\theta}$ (there is an explicit factor $\exp -(i\lambda_i^a\theta)$ in the numerator for every positive λ). Hence $H^{0,k}(\theta)$, for $k > 0$, contains only strictly negative powers of $e^{i\theta}$.

Combining these bits of information, we see that all Fourier coefficients of $H^{0,k}$ vanish, so $H^{0,k} = 0$ for $k \neq 0$. The same type of reasoning, applied to $H^{0,0}$, shows that $H^{0,0}(\theta)$ is independent of θ, and equal to the number of zeros of V at which all λ_i^a are positive.

$H^{m,k}$ for non-zero m and k can be studied analogously. Let Ω^m be the bundle of differential forms of type (m,0). $H^{m,k}$ is the k^{th} $\bar{\partial}$ cohomology group with coefficients in Ω^m. Studying inequalities (5) and (7) with $E = \Omega^m$, one finds that for any power of t except t^m, the left-hand side of (5) has only strictly positive powers of $e^{i\theta}$ while the left-hand side of (7) has only strictly negative powers of $e^{i\theta}$. Hence $H^{m,k} = 0$ for $k \neq m$. For $k = m$, we learn, instead, that $H^{m,m}(\theta)$ is independent of θ and equal to the number of zeros of V at which precisely m of the λ_i^a are negative.

Finally, let us discuss to what extent the assumptions made in this paper can be relaxed.

It is not clear whether inequalities analogous to (5) and (7) hold for holomorphic vector fields that do not generate isometries of any Kahler metric. If they do hold, the methods of this paper do not appear to be suitable for proving them.

If one does assume that V generates an isometry, our further requirement that V should generate a U(1) action is clearly not essential. If it were removed, the above discussion would change only in that we would not have a simple expansion in integral powers of $e^{i\theta}$. In any case, little is lost by requiring V to be a U(1) generator, since any generator of an isometry is a linear combination of U(1) generators.

Finally, we should discuss one important assumption made in the above discussion which can be eliminated. This is the assumption that V has only isolated zeros. It plays a simplifying role analogous to the role of the assumption in ordinary Morse theory that a function has only simple zeros, but it is not essential.

Suppose that the zeros of V are a complex manifold N, with connected components N_a. For large s, the low-lying eigenfunctions of Δ_s are concentrated near the N_i. The large s behavior of the spectrum of Δ_s can be determined by means of the Born-Oppenheimer approximation,[5] analogously to the treatment of degenerate Morse theory in reference (2). We will discuss this matter only briefly.

Let us focus on the contribution to the generalization of equation (5) of one component N_a of the space of zeros of V. Near N_a, in the directions orthogonal to N_a, V behaves as $\lambda_a^i z^i \frac{\partial}{\partial z^i}$, with certain integers λ_a^i that are constant on N_a. Let n_a be the number of negative λ_a^i. We may think of n_a as the index of N_a. Then, by study of the large s limit of Δ_s, it may be shown that the left-hand side of (5) must be replaced by

$$\sum_a t^{n_a} \sum_k t^k H^k (\theta; N_a; E_a) \tag{36}$$

Here $H^k (N_a; E_a)$ is the k^{th} $\bar{\partial}$ cohomology group of N_a, with coefficients in a bundle E_a that must now be described.

In a small neighborhood of N_a, Δ_s can be written approximately (the approximation is good enough for large s) as

$$\Delta_s = \Delta_{N_a} + \Delta_s^T \tag{37}$$

Here Δ_{N_a} is the usual Laplacian $\bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$ of N_a, and Δ_s^T acts in the directions orthogonal to N_a.

Choosing a tubular neighborhood $M(N_a)$ of N_a and projecting each point to the point in N_a to which it is nearest, $M(N_a)$ has the structure of a fiber bundle over N_a. Given a point x in N_a, one may restrict Δ_s^T to an operator Δ_s^x acting on the fiber over x in $M(N_a)$.

Let V_x be the vector space consisting of all eigenfunctions of Δ_s^x for which the eigenvalue vanishes in the leading large s approximation. Then V_x varies holomorphically with x, giving a holomorphic bundle Q_a over N_a. (Actually V_x and Q_a are infinite dimensional. The discussion can be made finite dimensional by fixing a representation of the circle group generated by V.) Q_a inherits from M a holomorphic action of V.

Then $E_a = E \otimes Q_a$ is the bundle that appears in (36). Equation (36) follows by remarks analogous to those that entered the discussion of degenerate Morse theory in reference (2).

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