Field Theory With a Vector Global Symmetry

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arXiv:1909.1054

arXiv:191?.????

Introduction

- Motivated by recent advances with fractons, but not sure this talk is relevant to fractons.
- Study global symmetries, whose conserved charge is a vector.
- One example is the momentum $P^i = \int_{space} T_0^i$. Its conserved Noether current $T^{\mu\nu}$ is symmetric.
- Oneform symmetry is common in relativistic field theories.
- We'll present a hierarchical set of three symmetries, starting with the most special one and generalizing it.
- For each symmetry (for simplicity $U(1)$) we will present:
  - the conserved currents
  - a class of field theories with the symmetry
  - a concrete example based on a scalar field theory

Relativistic oneform global symmetry

A lot of earlier work, here we follow [Gaiotto, Kapustin, NS, Willett]

$$\partial_\mu J^{[\mu\nu]} = 0$$

Charges

$$\mathcal{Q}(\mathcal{C}) = \int_{\mathcal{C}} J^{[\mu\nu]} n_{[\mu\nu]}$$

$\mathcal{C}$ a codimension 2 manifold in spacetime orthogonal to $n_{[\mu\nu]}$.

$\mathcal{Q}$ is topological – it does not change under small deformations of $\mathcal{C}$. Specifically, it does not change unless $\mathcal{C}$ crosses another operator.

Relativistic oneform global symmetry

In nonrelativistic terms

$$\partial_0 J_0^j - \partial_i J^{[ij]} = 0$$
$$G = \partial_i J_0^i = 0$$

Charges at fixed time

$$\mathcal{Q}(\mathcal{C}) = \int_{\mathcal{C}} J_0^j n_j$$

$\mathcal{C}$ a codimension 1 manifold in space orthogonal to n_j.

$\mathcal{Q}$ is topological – it does not change under small changes of $\mathcal{C}$.

Relativistic oneform global symmetry

$$\partial_0 J_0^j - \partial_i J^{[ij]} = 0$$
$$G = \partial_i J_0^i = 0$$

Example:

Maxwell theory – $U(1)$ gauge theory

$$J_0^j = F_0^j$$
$$J^{[ij]} = F^{[ij]}$$

$G = 0$ is Gauss law.

$Q(\mathcal{C})$ is the electric flux through $\mathcal{C}$.

The charged operators are Wilson lines.

There is also a magnetic symmetry, but we will not discuss it here.

Relativistic oneform global symmetry

Example: $U(1)$ lattice gauge theory

p, l, s are plaquettes, links, and sites

In $A_0 = 0$ the Hamiltonian is

$$\mathcal{H} = \sum_p \left(U_p + U_p^+\right) + \sum_l E_l^2$$

The variables are $U(1)$ elements U_l and their conjugate momenta E_l. $U_p = \prod_{l \subset p} U_l$ (oriented product)

Impose Gauss law $G = \sum_{l \supset s} E_l = 0$ (oriented sum)

$Q(\mathcal{C}) = \sum_{l \subset \mathcal{C}} E_l$ (oriented sum over the links pierced by $\mathcal{C}$) is the electric flux through $\mathcal{C}$. It is conserved and topological.

Nonrelativistic oneform global symmetry

As before,

$$\partial_0 J_0^j - \partial_i J^{[ij]} = 0 ,$$

but now we do not impose $G = \partial_i J_0^i = 0$.

The conservation leads to $\partial_0 G = \partial_i \partial_0 J_0^i = 0$,

i.e. G is conserved at every point, but it is not zero.

As before, conserved charges at fixed time

$$\mathcal{Q}(\mathcal{C}) = \int_{\mathcal{C}} J_0^j n_j ,$$

but now $\mathcal{Q}$ is not topological.

A cruder charge is $Q^j = \int_{space} J_0^j$.

Correspondingly, point operators can transform under $\mathcal{Q}(\mathcal{C})$, Q^j.

Nonrelativistic oneform global symmetry

$$\partial_0 J_0^j - \partial_i J^{[ij]} = 0$$

Therefore $$\partial_0 G = \partial_i \partial_0 J_0^i = 0$$

$$\mathcal{Q}(\mathcal{C}) = \int_{\mathcal{C}} J_0^j n_j$$

If there is an $\mathcal{M}$, such that $\mathcal{C} = \partial\mathcal{M}$

$$\mathcal{Q}(\mathcal{C}) = \int_{\mathcal{M}} \partial_i J_0^i = \int_{\mathcal{M}} G$$

The conservation of $\mathcal{Q}(\mathcal{C})$ implies the conservation of G.

But the conservation of $\mathcal{Q}(\mathcal{C})$ contains more information, because $\mathcal{C}$ might not be a boundary.

Nonrelativistic oneform global symmetry

Lattice example:

$U(1)$ lattice gauge theory in $A_0 = 0$, but without imposing Gauss law $G = \sum_{l\supset s} E_l = 0$ [Kitaev]

$$\mathcal{H} = \sum_p \left(U_p + U_p^+\right) + \sum_l E_l^2 + \sum_s \left(\sum_{l\supset s} E_l\right)^2$$

The last term imposes Gauss law energetically.

Interpret as $U(1)$ gauge theory with charged matter at the lattice scale.

$G = \sum_{l\supset s} E_l$, is conserved, but it is nonzero.

$Q(\mathcal{C}) = \sum_{l\subset \mathcal{C}} E_l$ are conserved, but they are not topological.

Related discussion in [Hermele, Fisher, Balents; Williamson, Bi, Cheng].

Nonrelativistic oneform global symmetry

A class of continuum examples

$$\partial_0 J_0^j - \partial_i J^{[ij]} = 0$$

Couple the $U(1)$ gauge theory to charged matter fields, such that we still have

$$J_0^j = F_0^j$$
$$J^{[ij]} = F^{[ij]}$$

For that, the matter theory should have an operator $\mathcal{O}$ satisfying $\partial_0 \mathcal{O} = 0$.

The $U(1)$ gauge theory couples to $\left(J_0^{matter} = \mathcal{O}, J^{i\,matter} = 0\right)$

$$\mathcal{L}_1 = \mathcal{L}_0 + \left(F_0^i\right)^2 - \left(F^{ij}\right)^2 + A_0 \mathcal{O}$$

with $\mathcal{L}_0$ the matter Lagrangian. This is $U(1)$ gauge invariant and has the nonrelativistic oneform symmetry with $G = \partial_i J_0^i = \mathcal{O}$.

Nonrelativistic oneform global symmetry

A class of continuum examples

$$\partial_0 \mathcal{O} = 0$$

Such a matter theory has infinitely many conserved charges.

$C(x^i)\mathcal{O}$ is conserved for every $C(x^i)$.

The charged matter is not mobile.

A concrete example:

A complex scalar field Φ with

$$\mathcal{L}_0 = i\bar{\Phi}\partial_0\Phi - \partial_i(\bar{\Phi}\Phi)\partial^i(\bar{\Phi}\Phi) \; - |\Phi|^4 + \cdots$$

No $\partial_i\bar{\Phi}\partial^i\Phi$ term. Highly nonstandard. Similar to fractons.

Invariance under $\Phi \to e^{iC(x^i)}\Phi$

The charge density $\mathcal{O} = |\Phi|^2$ at a point is conserved.

A more general vector symmetry

Previous case $\partial_0 J_0^j - \partial_i J^{ij} = 0$

with antisymmetric J^{ij}.

Generalize to J^{ij} with no restriction on the symmetry of ij.

Now $\partial_0 G = \partial_i \partial_0 J_0^i = \partial_i \partial_j J^{ij} \neq 0$.

Therefore, the charge operators on codimension one manifold $\mathcal{C}$ $Q(\mathcal{C}) = \int_{\mathcal{C}} J_0^j n_j$ are no longer conserved, but the cruder charge

$$Q^j = \int_{space} J_0^j$$

is conserved.

A more general vector symmetry

A class of continuum examples

$$\partial_0 J_0^j - \partial_i J^{ij} = 0$$

For $J^{(ij)} = 0$, we coupled charged matter with $\partial_0 \mathcal{O} = 0$ to a $U(1)$ gauge field.

Now, we take a matter theory with

$$\partial_0 \mathcal{O} - \partial_i \partial_j \mathcal{O}^{ij} = 0 \qquad \text{(with } \mathcal{O}^{(ij)}\text{)}$$

Couple the $U(1)$ gauge field to $\left(J_0^{matter} = \mathcal{O},\ J^{i\,matter} = \partial_j \mathcal{O}^{ij}\right)$

$$\mathcal{L}_1 = \mathcal{L}_0 + \left(F_0^i\right)^2 - \left(F^{ij}\right)^2 + A_0 \mathcal{O} - A_i \partial_j \mathcal{O}^{ij}$$

The conserved current of the global symmetry are

$$J_0^j = F_0^j$$
$$J^{ij} = F^{[ij]} - \mathcal{O}^{ij}$$

A more general vector symmetry

A class of continuum examples

$$\partial_0 \mathcal{O} - \partial_i \partial_j \mathcal{O}^{ij} = 0$$

Such a matter theory (before coupling to the gauge field) has the conserved charges

$$Q = \int_{space} J_0^{matter} = \int_{space} \mathcal{O}$$

$$Q^j = \int_{space} x^j J_0^{matter} = \int_{space} x^j \mathcal{O}$$

They can be interpreted as:

- a global $U(1)$ symmetry (which we gauge)
- a vector symmetry, dipole symmetry, with charge Q^j

After the gauging we are left only with the vector symmetry.

A more general vector symmetry

A class of continuum examples

$$\partial_0 \mathcal{O} - \partial_i \partial_j \mathcal{O}^{ij} = 0$$

This defining equation is local and therefore, the discussion makes sense on every manifold.

This is not true for the charge, $Q^j = \int_{space} x^j \mathcal{O}$, which makes sense only on $\mathbb{R}^D$.

After coupling to the $U(1)$ gauge field the conserved current of the vector symmetry is

$$J_0^j = F_0^j$$
$$J^{[ij]} = F^{[ij]} - \mathcal{O}^{ij}$$

No explicit x^j dependence.

A more general vector symmetry

A concrete continuum example [Pretko]

$$\partial_0 \mathcal{O} - \partial_i \partial_j \mathcal{O}^{ij} = 0$$

A complex scalar field Φ with

$$\mathcal{L}_0 = i\bar{\Phi}\partial_0\Phi - \partial_i(\bar{\Phi}\Phi)\partial^i(\bar{\Phi}\Phi) - |\Phi|^4$$
$$+ i\left(\bar{\Phi}^2\partial_i\Phi\partial^i\Phi - \Phi^2\partial_i\bar{\Phi}\partial^i\bar{\Phi}\right) + \cdots$$

Again, no $\partial_i\bar{\Phi}\partial^i\Phi$ term.

The new term $i\left(\bar{\Phi}^2\partial_i\Phi\partial^i\Phi - \Phi^2\partial_i\bar{\Phi}\partial^i\bar{\Phi}\right)$ breaks the symmetry $\Phi \to e^{iC(x^i)}\Phi$ to $\Phi \to e^{i\alpha + ic_i x^i}\Phi$.

Here

$$\mathcal{O} = |\Phi|^2$$
$$\mathcal{O}^{ij} = |\Phi|^4\delta^{ij} + \cdots$$

Higher order terms can lead to a traceless symmetric tensor in $\mathcal{O}^{ij}$.

Summary of the symmetries

$$\partial_0 J_0^j - \partial_i J^{ij} = 0$$

- A vector symmetry: J^{ij} not restricted

  The charge $$Q^j = \int_{space} J_0^j$$

- Nonrelativistic oneform symmetry: impose also

$$J^{ij} = -J^{ji}$$

$\mathcal{Q}(\mathcal{C}) = \int_{\mathcal{C}} J_0^j n_j$ is associated with a nontopological manifold $\mathcal{C}$

- As in the relativistic symmetry: impose also

$$\partial_j J_0^j = 0$$

$\mathcal{Q}(\mathcal{C}) = \int_{\mathcal{C}} J_0^j n_j$ is associated with a topological manifold $\mathcal{C}$

Gauging

Start with a theory with $\partial_0 J_0^j - \partial_i J^{ij} = 0.$

We gauge by introducing sources $B_{0j}, \mathcal{A}_{ij} = \frac{1}{2}(B_{[ij]} + S_{(ij)}).$

(No need to introduce $S_{(ij)}$ when $J^{(ij)} = 0$.)

Add to the Lagrangian the minimal coupling terms

$$B_{0j} J_0^j - \mathcal{A}_{ij} J^{ij}$$

Invariance under the gauge symmetry

$$B_{0i} \to B_{0i} + \partial_0 c_i$$
$$\mathcal{A}_{ij} \to \mathcal{A}_{ij} + \partial_i c_j$$

- No c_0
- $\mathcal{A}_{ij}$ has no symmetry in ij – peculiar gauge field
- c_i is not a $U(1)$ gauge field

Gauging

$$B_{0i} \to B_{0i} + \partial_0 c_i$$
$$\mathcal{A}_{ij} \to \mathcal{A}_{ij} + \partial_i c_j$$

We can also add kinetic terms for these fields using the gauge invariant field strengths

$$\epsilon_{ij} = \partial_0 \mathcal{A}_{ij} - \partial_i B_{0j}$$
$$\beta_{ijk} = \partial_i \mathcal{A}_{jk} - \partial_j \mathcal{A}_{ik}$$

The standard $H_{\mu\nu\rho}$ are linear combinations of them

$$H_{0ij} = \epsilon_{ij} - \epsilon_{ji}$$
$$H_{ijk} = \beta_{ijk} - \beta_{ikj} + \beta_{jki}$$

Gauging

Let us implement it explicitly in the $U(1)$ theory coupled to a matter system. We started with a matter theory with

$$\partial_0\mathcal{O} - \partial_i\partial_j\mathcal{O}^{ij} = 0$$

Above we gauged an ordinary $U(1)$ with

$(J_0^{matter} = \mathcal{O},\ J^{i\,matter} = \partial_j\mathcal{O}^{ij})$, i.e.

$$\mathcal{L}_1 = \mathcal{L}_0 + \left(F_0^i\right)^2 - \left(F^{ij}\right)^2 + A_0\mathcal{O} - A_i\partial_j\mathcal{O}^{ij}$$

The remaining global symmetry

$$J_0^j = F_0^j$$
$$J^{[ij]} = F^{[ij]} - \mathcal{O}^{ij}$$

Now, gauge this remaining vector global symmetry.

Gauging

$$J_0^j = F_0^j$$
$$J^{ij} = F^{[ij]} - \mathcal{O}^{(ij)}$$

Add new gauge fields $B_{0j}, \mathcal{A}_{ij} = \frac{1}{2}(B_{[ij]} + S_{(ij)})$ with minimal coupling

$$\mathcal{L}_2 = \mathcal{L}_0 + \left(F_0^i + B_0^i\right)^2 - \left(F^{ij} + B^{ij}\right)^2 + A_0\mathcal{O} + A_{ij}\mathcal{O}^{ij} + \cdots$$

where $A_{ij} = \frac{1}{2}\left(S_{ij} + \partial_i A_j + \partial_j A_i\right)$.

The $U(1)$ gauge symmetry acts as

$$A_0 \to A_0 + \partial_0 \alpha$$
$$A_i \to A_i + \partial_i \alpha$$
$$A_{ij} \to A_{ij} + \partial_i \partial_j \alpha$$

In addition, we also have the gauge symmetry…

Gauging

Additional gauge symmetry

$$A_i \to A_i + c_i$$
$$B_{0i} \to B_{0i} + \partial_0 c_i$$
$$B_{ij} \to B_{ij} + \partial_i c_j - \partial_j c_i$$

Note, $A_{ij} = \frac{1}{2}\left(S_{ij} + \ \partial_i A_j + \partial_j A_i\right)$ is invariant under c_i

Can add kinetic terms using

$$H_{0ij} = \partial_0 B_{ij} - \partial_i B_{0j} + \partial_j B_{0i}$$
$$H_{ijk} = \partial_i B_{jk} - \partial_j B_{ik} + \partial_k B_{ij}$$
$$\mathcal{E}_{ij} = \partial_0 A_{ij} \ - \partial_i \partial_j A_0$$
$$\mathcal{B}_{ijk} = \partial_i A_{jk} - \partial_j A_{ik}$$

A_i is "Higgsed" and is lifted (becomes massive) with B_{i0} and B_{ij}.

We are left with A_0 and A_{ij}.

Gauging

We can do the gauging in one step. (Similar to discussions in [Rasmussen, You, Xu; Pretko;Slagle, Prem, Pretko;…].)

We started with a matter theory with

$$\partial_0 \mathcal{O} - \partial_i \partial_j \mathcal{O}^{ij} = 0$$

We couple these operators to sources

$$\mathcal{L}_0 + A_0 \mathcal{O} + A_{ij} \mathcal{O}^{ij} + \cdots$$

Hence, there is a $U(1)$ gauge symmetry

$$A_0 \to A_0 + \partial_0 \alpha$$
$$A_{ij} \to A_{ij} + \partial_i \partial_j \alpha$$

And we can add kinetic terms using

$$\mathcal{E}_{ij} = \partial_0 A_{ij} - \partial_i \partial_j A_0$$
$$\mathcal{B}_{ijk} = \partial_i A_{jk} - \partial_j A_{ik}$$

Summary

- A hierarchy of global symmetries, whose charges Q^i carry a spatial vector index.
- For every one of these we presented a large class of theories exhibiting them.
  - All these examples are based on a $U(1)$ gauge theory coupled to a special matter theory
  - We showed concrete examples of these matter theories
- We can gauge these new symmetries. The needed gauge field is an antisymmetric tensor $B_{[\mu\nu]}$ and in the general symmetry we also need a symmetric tensor $S_{(ij)}$.