

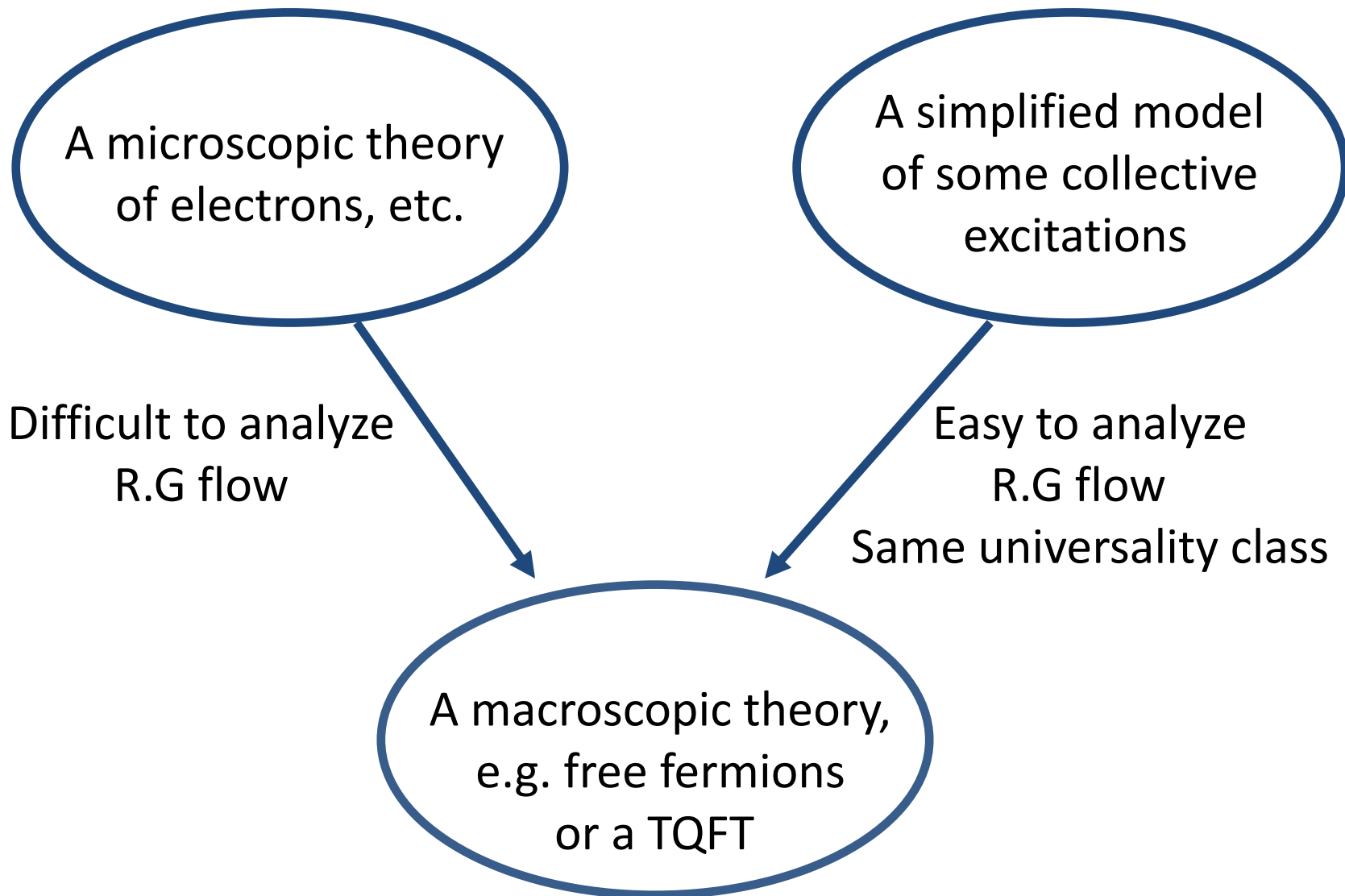
Gapped Boundary Phases of Topological Insulators via Weak Coupling

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Nathan Seiberg and Edward Witten, [arXiv:1602.04251](https://arxiv.org/abs/1602.04251)

Often in CM Physics



The Three Models

- The macroscopic model gives an exact description of the long distance physics.
- The simplified model flows to the macroscopic model and captures some features of the higher energy modes (e.g. properties of some quasi-particles), but it is not exact.
- The simplified model is particularly useful when it is weakly coupled. We can vary parameters across phase transitions, while keeping \hbar small. Semi-classical methods are reliable.
- Powerful consistency conditions based on anomalies can constrain the search for these models [['t Hooft](#)].

Phases of Theories (Long Distance Behavior)

- Gapless (= massless) $\lim_{x \rightarrow \infty} \langle \phi(x) \dots \rangle \sim \frac{1}{|x|^{2\Delta}}$
 - Free theory (trivial)
 - Interacting = nontrivial conformal field theory (CFT)
 - Gapped $\lim_{x \rightarrow \infty} \langle \phi(x) \dots \rangle \sim e^{-a|x|}$
 - Trivial bulk theory
 - Trivial boundary
 - Gapless boundary modes
 - Gapped TQFT on the boundary
 - Nontrivial bulk topological quantum theory (TQFT)
 - Same as above
- Bulk is not completely trivial. Symmetry Protected Topological (SPT) phase

Topological Insulators [Kane, Mele; ...]

- Insulator
 - Unbroken global $U(1)_A$. The electromagnetic gauge field A can be viewed as a classical background field.
 - Gapped and trivial bulk
- Assume it is time-reversal (T) invariant
- Nontrivial boundary
 - Can be understood in examples, more generally follows from anomalies
 - Typically, massless fermions (gapless)
 - Can also lift the fermions and have gapped boundary states. (Examples by [Metlitski, Kane, Fisher; Wang, Potter, Senthil...].)

Topological Insulator: Simple Example

Massive electron with positive real mass m outside the material and mass $-m$ inside the material.

- Since m is real, it preserves $U(1)_A$ and T (and also charge conjugation C , but this is not a symmetry in CM physics).
- As the mass varies in space, it leads to massless (gapless) fermions on the boundary [Jackiw, Rebbi].
- Perform a chiral rotation inside the material and have positive mass everywhere. Now $\theta = \pi$ inside and $\theta = 0$ outside [Qi, Hughes, Zhang; Essin, Moore, Vanderbilt]:

$$\frac{1}{8\pi} \int \left(F \wedge F + \frac{1}{24} \text{Tr} R \wedge R \right)$$

In most of the talk we will neglect the gravitational term.

Topological Insulator: simple example

Could start with $\frac{1}{8\pi} \int F \wedge F$ inside the material, but not outside.

Another perspective on the boundary modes:

2+1-dimensional complex massless fermions have “parity anomaly.”

We would like to preserve $U(1)_A$ and T . But we can preserve

- either $U(1)_A$ and violate T
- or T and violate $U(1)_A$
- or $U(1)_A$ and T , but the theory is not truly 2 + 1-dimensional. It needs a bulk interaction. This is an example of anomaly inflow [Callan, Harvey].

Massless boundary modes are associated with $U(1)_A$ and T . They are robust.

Digression: The Parity Anomaly

Consider a single complex fermion coupled to a gauge field A .

Its partition function is [Alvarez-Gaumé, Della Pietra Moore]

$$Z = |\text{Det } \mathcal{D}| \exp\left(\pm \frac{i\pi\eta}{2}\right)$$

$$\eta(A) = \lim_{\epsilon \rightarrow 0^+} \sum_i \exp(-\epsilon |\lambda_i|) \text{sign}(\lambda_i)$$

The sign ambiguity reflects the T -breaking.

Cannot write the phase as $\exp\left(\pm \frac{i}{8\pi} \int AdA\right)$ because this Chern-Simons term is not well defined

$$\exp\left(\pm \frac{i\pi\eta}{2}\right) = \pm \exp\left(\pm \frac{i}{8\pi} \int AdA + \text{gravitational term}\right)$$

Return to the Topological Insulator

Recall the APS theorem

$$\text{Index}(\mathcal{D}) = \frac{1}{8\pi^2} \int_{\text{Bulk}} \left(F \wedge F + \frac{1}{24} \text{tr} R \wedge R \right) - \frac{\eta}{2}$$

Therefore, we can define [Witten]

$$Z \equiv (-1)^{\text{Index}(\mathcal{D})} |\text{Det } \mathcal{D}| =$$

$$|\text{Det } \mathcal{D}| \exp\left(\pm \frac{i\pi\eta}{2}\right) \exp\left(\mp \frac{i}{8\pi} \int_{\text{Bulk}} \left(F \wedge F + \frac{1}{24} \text{tr} R \wedge R \right)\right)$$

It is T -invariant (real) but depends on the bulk.

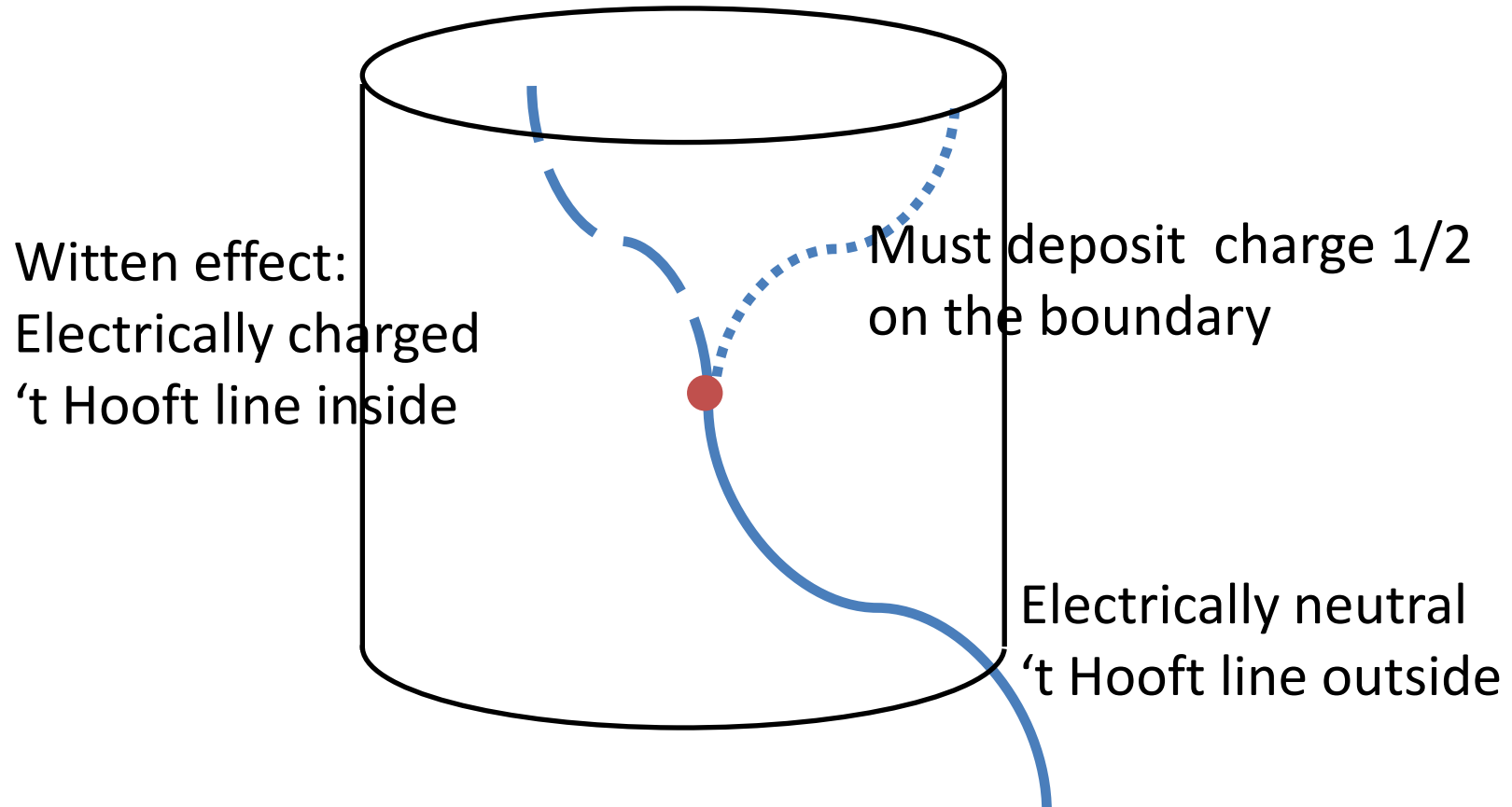
This is the partition function of the topological insulator.

Topological Insulator

Start with $\frac{1}{8\pi} \int F \wedge F$ inside the material, but not outside.

Another perspective on “nontrivial physics on the boundary.”

Bring a neutral magnetic monopole through the boundary



Topological Insulator

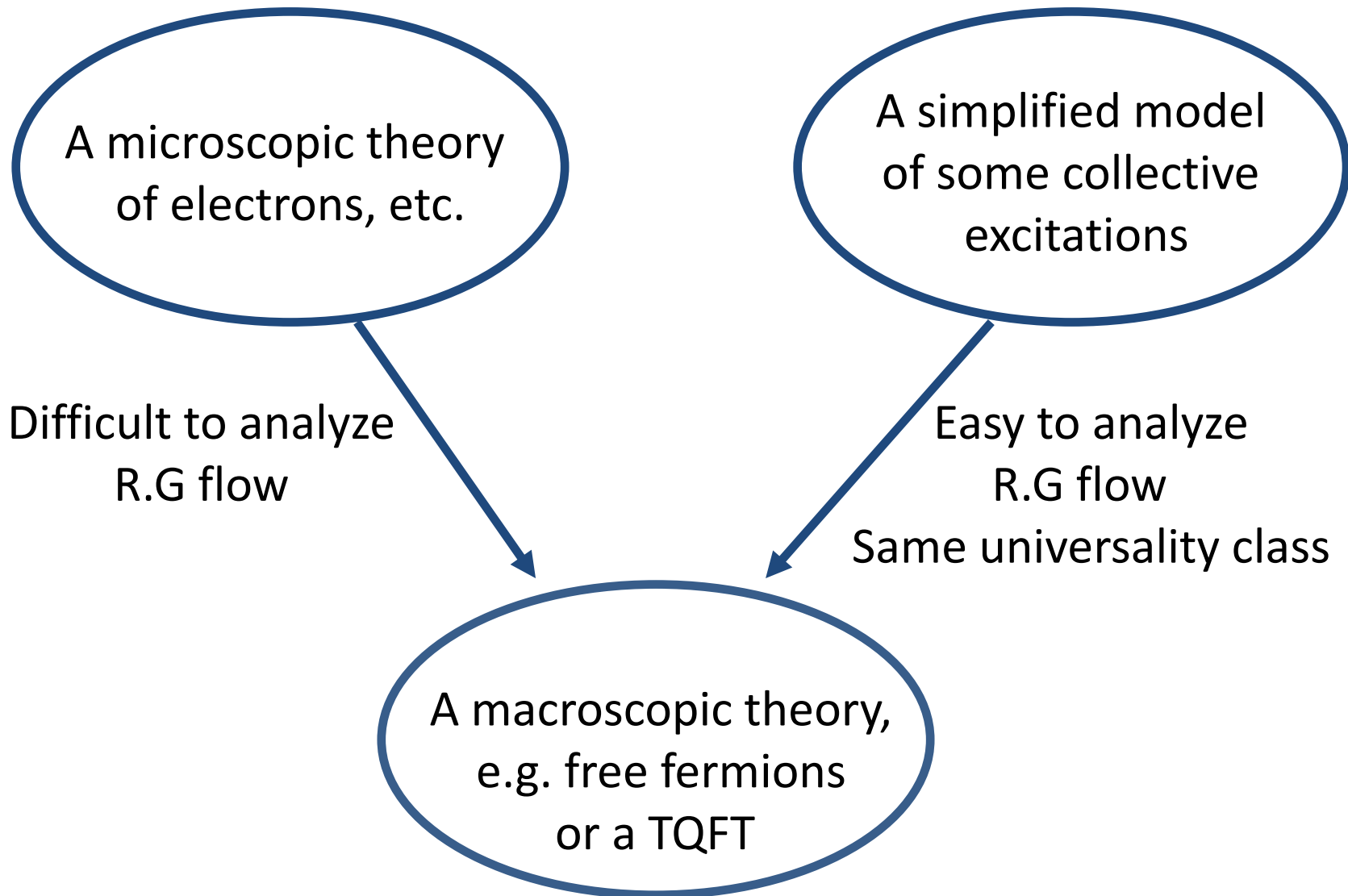
Start with $\frac{1}{8\pi} \int F \wedge F$ inside the material, but not outside.

Massless boundary modes are associated with $U(1)_A$ and T .

They are robust – cannot be lifted by small perturbations.

- Can we add a large perturbation and gap the system?
- Something must remain on the boundary to account for the anomaly inflow.
- Can there be a TQFT on the boundary with the same anomaly? (Examples by [Metlitski, Kane, Fisher; Wang, Potter, Senthil...].)
- Not obvious whether a given TQFT has the right anomaly. (Particularly challenging to control the gravitational anomaly.)

Often in CM Physics



The Simplified Model

We will present a simplified model for a topological insulator, which extends the free fermion model above.

It will allow us to gap the boundary in a controlled way.

\hbar will remain arbitrarily small. So weak coupling analysis is reliable.

The Simplified Model

Extend the model with a single massless boundary fermion.

- Emergent $U(1)_a$ gauge field on the boundary
- Scalar w of $U(1)_a$ charge 1, which can Higgs it to be trivial.
- Massless fermion χ with $U(1)_A \times U(1)_a$ charges $(1, 2s)$
 - For integer s no additional anomaly associated with a .
 - Below we will argue that s has to be even.

In a phase with $\langle w \rangle \neq 0$ the low-energy spectrum consists of a massless fermion with $U(1)_A$ charge one.

So this system contains the previous system – same anomaly.
(Even the same gravitational anomalies.)

The Simplified Model

Add:

- Scalar Φ with $U(1)_A \times U(1)_a$ charges $(2, 4s)$ such that we can have a T -invariant coupling $\chi\chi\Phi^* + \text{c.c.}$

In a phase with $\langle w \rangle = 0$, but $\langle \Phi \rangle \neq 0$ the theory is gapped:

- Higgs $U(1)_a \rightarrow \mathbf{Z}_{4s}$. No massless gauge field.
- χ acquires a mass from $\chi\chi\Phi^*$
- Unbroken T and global $U(1)_A$ symmetry (linear combination of the original global $U(1)_A$ and gauge $U(1)_a$)
- Our system has the right anomaly to be a boundary state.
- It has a gapped boundary phase with a TQFT.
- Everything can be analyzed explicitly.

The Massive Spectrum (Quasiparticles)

- w quanta are $U(1)_A$ neutral bosons transforming with “charge” 1 under \mathbf{Z}_{4s} .
- χ quanta are $U(1)_A$ neutral fermions transforming with “charge” $2s$ under \mathbf{Z}_{4s} .
- Interesting spectrum of vortices from $U(1)_a \rightarrow \mathbf{Z}_{4s}$:
 - The elementary vortex (vorticity $\nu = \pm 1$) has a single χ zero mode. It exhibits non-Abelian statistics.
 - More generally, all odd ν vortices have non-Abelian statistics.
 - Even ν vortices have Abelian statistics.

Monopole Operators

- As every 2+1-dimensional $U(1)_a$ gauge theory, our system has a global $U(1)_J$ symmetry, whose current is $\frac{1}{2\pi} da$.
- The local operators charged under it are monopole operators. They correspond to vortices with $\nu = 4s$.
- Since the microscopic system of electrons does not have this $U(1)_J$ symmetry, we would like to add the monopole operator to the Lagrangian (Hamiltonian).
- This can be done while keeping the coefficient of the monopole operator and \hbar small. Preserve weak coupling.

Monopole Operators

- Must be able to add a monopole operator to the Lagrangian (Hamiltonian).
- Quantizing the monopole operator we find that a $U(1)_A \times U(1)_a$ neutral operator has spin $\frac{s}{2} \bmod \mathbf{Z}$.
- Therefore, we can add it to the Lagrangian only for s even.
- This is a symptom of a more general phenomenon...

Spin/Charge Relation

Consider a system of electrons with an arbitrary Hamiltonian such that the nuclear spins are not important.

All the states of the system (in finite volume) and all the local operators must satisfy a selection rule:

$$2 \text{ Spin} = U(1)_A \text{ Charge mod } 2$$

This is a powerful constraint on any long distance description (and any simplified model that flows to it).

One way of thinking about the spin/charge constraint is similar to 't Hooft anomaly constraints...

Spin/Charge Relation

't Hooft coupled a system with a global symmetry to a background gauge field and tracked the anomalies to long distances.

Similarly, we couple the microscopic system to a background metric and a background A .

Naively, since we have spinors, we need the background manifold to have a spin structure.

But, we can also place the system on a non-spin manifold by letting A be a $Spin_c$ connection...

Spin/Charge Relation

We can also place the system on a non-spin manifold by letting A be a $Spin_c$ connection.

(This means that the obstruction to spin structure is corrected by A not being a $U(1)$ gauge field; i.e. $\int \frac{F}{2\pi} = \frac{1}{2} \int w_2 \pmod{\mathbf{Z}}$.)

Note, we are not really interested in the behavior of our system of electrons in this more general background. This is merely a device to constrain the long distance behavior .

Therefore, the macroscopic theory should also satisfy this relation. And if we study it using a simplified model that flows to it, this model should also satisfy it.

Spin/Charge Relation

Back to our model with an emergent $U(1)_a$ gauge field and

- scalar w with $U(1)_A \times U(1)_a$ charges $(0, 1)$
- fermion χ with $U(1)_A \times U(1)_a$ charges $(1, 2s)$
- scalar Φ with $U(1)_A \times U(1)_a$ charges $(2, 4s)$

All the perturbative $U(1)_a$ invariant operators satisfy the spin/charge relation (fermions have odd $U(1)_A$ charges and bosons have even $U(1)_A$ charge).

For even s this is also true for the monopole operators.

For odd s the monopole operators do not satisfy it.

Spin/Charge Relation

For odd s the perturbative spectrum satisfies the relation. But monopole operators do not satisfy it.

This is a new anomaly.

It allows us to exclude models. In our case it forces s to be even.

The Low Energy TQFT

Integrate out the high energy modes and explicitly construct the low energy theory.

It is a TQFT

It can be described as a Chern-Simons theory in many ways, e.g. with gauge group (with levels that depend on s)

$$\frac{U(2) \times U(1)}{\mathbf{Z}_2} \times U(1)$$

The spins, statistics, and charges of the line observables reproduce the semi-classical results of the quasi-particles.

This TQFT with $s = 2$ in [\[Metlitski, Kane, Fisher\]](#).

Conclusions

- Topological phases of matter are interesting.
 - They exhibit rich phenomena. Some of them have already been encountered by high energy physicists, but most of them have not.
 - Mathematics, quantum field theory, condensed matter physics...
- We have presented a weakly coupled T -invariant theory with a global $U(1)_A$ symmetry. It has two interesting phases:
 - Massless charged fermions. Hence, the right anomaly to be the boundary of a topological insulator.
 - Gapped phase with a TQFT.
 - Explicit, calculable.

Conclusions

- The analysis of this system, despite being weakly coupled, has many interesting subtleties.
- New consistency conditions
- New anomalies
- More models
- Topological superconductors ($U(1)_A$ is broken to \mathbf{Z}_2)
- Many interesting questions