Exotic Global Symmetry

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NS, Shu-Heng Shao, arXiv:200?.?????

Introduction

- Fractons challenge our understanding of quantum field theory.
- They exhibit phenomena that are seemingly impossible in continuum quantum field theory.
- They also exhibit peculiar global symmetries, most of them were not seen before. (Recent work [NS; Qi, Radzihovsky, Hermele].)
- We will attempt to organize these global symmetries, without reference to the details of the underlying theory.
- We will focus on $U(1)$ global symmetries and discuss their Noether currents. This will lead us to their conserved charges and their properties.
- Similar conserved charges exist also for finite groups.

Qualifications

- Throughout most of the discussion, we will limit ourselves to flat spacetime with a spatial three-dimensional torus. We will freely raise and lower spatial indices.
- We will use continuum notation (can be done on the lattice)
  - Some of the symmetries we will discuss exist only in the continuum theory and not in the corresponding lattice theory.
  - However, to make sense of some of the continuum expressions below, one must place them on lattice. Some of the observables, fields, gauge transformation parameters, etc. will have to be discontinuous, e.g. some observables are different integers at different points.

Qualifications

- We will discuss various gauge theories with certain gauge parameters. We have not analyzed the global part of the group and the corresponding geometric structure. This is particularly challenging because the fields and the gauge parameters are in general discontinuous.

Ordinary $U(1)$ global symmetry

In a relativistic theory, a conserved Noether current

$$\partial_\mu J^\mu = 0$$

In nonrelativistic notation

$$\partial_0 J_0 = \partial_i J^i$$

The conserved charge

$$Q = \int_{space} J_0$$

Generalizations

J_0 is in a representation of the $SO(3)$ rotation group, or its cubic subgroup, $\boldsymbol{R_{time}}$. Denote J_0^I, e.g. vector J_0^i, tensor J_0^{ij}.

$$\partial_0 J_0^I = \partial_i J^{iI}$$

The spatial components of the current could be in various representations $\boldsymbol{R_{space}}$ (in the product of $\boldsymbol{R_{time}}$ and a vector).

The global symmetry charge is obtained by integrating J_0^I over the entire space or a closed subspace $\mathcal{C}$ (I is contracted with the integral measure)

$$Q^I = \int_{space} J_0^I \quad \text{or} \quad Q(\mathcal{C}) = \oint_{\mathcal{C}} J_0$$

The subspace $\mathcal{C}$ is such that the charge is conserved

$$\partial_0 Q(\mathcal{C}) = \oint_{\mathcal{C}} \partial_0 J_0 = \oint_{\mathcal{C}} \partial_i J^i = 0.$$

Generalizations

$$\partial_0 J_0^I = \partial_i J^{iI}$$
$$Q(\mathcal{C}) = \oint_{\mathcal{C}} J_0$$

Often, differential conditions, e.g. $\partial_i J_0^i = 0$, restrict the dependence on $\mathcal{C}$. $Q(\mathcal{C})$ could be independent of certain changes in $\mathcal{C}$ or even completely topological.

Algebraically, this condition performs a quotient of the space of charges.

Sometimes J^{iI} can be expressed as a total derivative such that

$$\partial_0 J_0^I = \partial_i \partial_j \cdots J^{ij\cdots I}$$

These are multipole symmetries.

Vector symmetry

$\boldsymbol{R_{time}} = \mathbf{3}$

$$\partial_0 J_0^i = \partial_j J^{ji}$$

$\boldsymbol{R_{space}}$ can be in $\mathbf{1} \oplus \mathbf{3} \oplus \mathbf{5}$ of $SO(3)$ or in a subset of them, or in a representation of its cubic subgroup. (More below.)

For $\boldsymbol{R_{space}} = \mathbf{1}$

$$\partial_0 J_0^i = \partial_i J$$

The conserved charge is

$$Q(\mathcal{C}) = \oint_{\mathcal{C}} J_0^i dx^i$$

with a one-dimensional $\mathcal{C}$.

Vector symmetry

$(\boldsymbol{R_{time}}, \boldsymbol{R_{space}}) = (\mathbf{3}, \mathbf{1})$

$$\partial_0 J_0^i = \partial_i J$$

$$Q(\mathcal{C}) = \oint_{\mathcal{C}} J_0^i dx^i$$

Example: a compact boson (not on the lattice) $\phi \sim \phi + 2\pi$

The current $J_0^i = \partial^i \phi, \;\; J = \partial_0 \phi$ is trivially conserved and the charge

$$Q(\mathcal{C}) = \oint_{\mathcal{C}} \partial_i \phi dx^i$$

is the winding number.

In this case the current satisfies a differential constraint $\partial^i J_0^j = \partial^j J_0^i$ making the dependence on $\mathcal{C}$ topological.

Vector symmetry

$\left(\boldsymbol{R_{time}}, \boldsymbol{R_{space}}\right) = (\mathbf{3}, \mathbf{3})$

$$\partial_0 J_0^i = \partial_j J^{[ji]}$$

The conserved charge is

$$Q(\mathcal{C}) = \oint_{\mathcal{C}} J_0^j n_j$$

$\mathcal{C}$ is a closed two-dimensional manifold (n_j is orthogonal $\mathcal{C}$).

If we also have the differential condition

$$\partial_i J_0^i = 0$$

the dependence on $\mathcal{C}$ is topological

These are the nonrelativistic and the relativistic one-form global symmetries of [Gaiotto, Kapustin, NS, Willett; NS].

Relativistic one-form global symmetry

$$\partial_0 J_0^i - \partial_j J^{[ji]} = 0$$
$$\partial_i J_0^i = 0$$

Example: Maxwell theory – $U(1)$ gauge theory (either in the continuum or on the lattice)

An electric symmetry – the charge is the electric flux

$$J_0^j \sim F_0^j = E^j$$
$$J^{[ij]} \sim F^{[ij]} \sim \epsilon^{ijk} B_k$$

The continuum theory (but not the lattice) also has a magnetic symmetry – the charge is the magnetic flux

$$J_0^i = B^i$$
$$J^{[ij]} = \epsilon^{ijk} E_k$$

A dipole global symmetry

Dipole global symmetry [Pretko] (with more indices – a multipole symmetry [Gromov])

$$\Phi \to e^{i\alpha + ic_i x^i}\Phi$$

Instead of its action on the fields, we characterize it by the conservation equation

$$\partial_0 J_0 = \partial_i \partial_j J^{ij}$$

$\boldsymbol{R_{time} = 1}$

J^{ij} is in $\boldsymbol{R_{space}} = \mathbf{1} \oplus \mathbf{5}$ of $SO(3)$.

This is a local equation and does not depend on the spatial manifold. (Note, on a torus the transformation $\Phi \to e^{i\alpha + ic_i x^i}\Phi$ makes sense only for special c_i.)

A dipole global symmetry

$$\partial_0 J_0 = \partial_i \partial_j J^{ij}$$

Ordinary charge $\qquad Q = \int_{space} J_0$

On $\mathbb{R}^3$ a dipole charge $\qquad Q^i = \int_{space} x^i J_0$

Similar to the local conservation of a symmetric $T^{\mu\nu}$ implying Lorentz invariance $\mathcal{M}^{\mu\nu} = \int_{space} (x^\mu T^{\nu 0} - x^\nu T^{\mu 0})$.

Example [Pretko]:

$$\mathcal{L} = (\partial_0 \phi)^2 - \left(\partial_i \partial^i \phi\right)^2$$

$$J_0 = \partial_0 \phi \ , \qquad J^{ij} = \partial^i \partial^j \phi$$

(Could study $\partial_0 J_0 = \partial_i \partial_j \partial^j J^i$ with $J^i = \partial^i \phi$, but this does not lead to more information.)

The cubic group (ignoring reflections)

Its representations are

- $\mathbf{1}$ – the trivial representation (a scalar)
- $\mathbf{3}$ – the vector representation V^i: V^x, V^y, V^z

The $SO(3)$ traceless symmetric tensor is decomposed as $\mathbf{3}' \oplus \mathbf{2}$

- $\mathbf{3}'$ – $T^{(ij)}$ with $i \neq j$: T^{xy}, T^{yz}, T^{zx}
- $\mathbf{2}$ – D^{ii} with vanishing trace: D^{xx}, D^{yy}, $D^{zz} = -D^{xx} - D^{yy}$. We will also label them as
  - $D^{[ij]k} = \epsilon^{ijk} D^{kk}$
  - $D^{i(jk)} = D^{[ij]k} + D^{[ik]j} = \epsilon^{ijk}\left(D^{kk} - D^{jj}\right)$
- **1'** – A. It arises in the three index symmetric tensor of $SO(3)$

A dipole global symmetry with cubic symmetry

$$\partial_0 J_0 = \partial_i \partial_j J^{ij}$$

Now, J^{ij} can be in different representations.

$\boldsymbol{R_{space}} = \mathbf{3}'$, i.e. $i \neq j$. The conserved charges are

$Q(\mathcal{C}) = \oint_{\mathcal{C}} J_0$ with $\mathcal{C}$ the (x, y), or (x, z), or (y, z) plane

$$Q^x(x) = \oint dydz\, J_0$$

and similarly for Q^y, Q^z.

This is a subsystem symmetry. (Not with full $SO(3)$ symmetry.)

In some lattice models (examples below) $Q^x(x)$ can have different integer values at different x. Seems impossible in continuum QFT.

A dipole global symmetry with cubic symmetry

$(\boldsymbol{R_{time}}, \boldsymbol{R_{space}}) = (\mathbf{1}, \mathbf{3}')$ $\partial_0 J_0 = \partial_i \partial_j J^{ij} \quad i \neq j$

Example:

$$\mathcal{L} = (\partial_t \phi)^2 - g_1 \left(\partial_i \partial^i \phi\right)^2 - g_2 \sum_{i \neq j} \left(\partial_i \partial_j \phi\right)^2$$

preserves the cubic symmetry.

- If $g_2 = 0$, full $SO(3)$ symmetry.
- If $g_1 = 0$, J^{ij} in $\boldsymbol{R_{space}} = \mathbf{3}'$ and leads to a subsystem symmetry [Slagle, Kim; Radičević]. The same system in $2+1$ dimensions in [Paramekanti, Balents, Fisher].

Gauging

$$\partial_0 J_0^I = \partial_i \partial_j \cdots J^{ij\cdots I}$$

I is an $\boldsymbol{R_{time}}$ index and $(ij \cdots I)$ are contracted to $\boldsymbol{R_{space}}$.

We couple to gauge fields

$$A_0 J_0 + AJ \qquad \text{(suppressed indices)}$$

Gauge invariance

$$A_0^I \to A_0^I + \partial_0 \lambda^I$$
$$A_{ij\cdots}^I \to A_{ij\cdots}^I + \partial_i \partial_j \cdots \lambda^I$$

A_0^I, λ^I in $\boldsymbol{R_{time}}$

$A_{ij\cdots}^I$ in $\boldsymbol{R_{space}}$

A differential condition on J_0^I means that some λ^I act trivially, i.e. λ^I is itself a gauge field (as in Cheeger-Simons differential characters).

Gauging the dipole symmetry

$$A_0 J_0 + A_{ij} J^{ij}$$

The gauge parameter λ is a scalar

$$A_0 \rightarrow A_0 + \partial_0 \lambda$$
$$A_{ij} \rightarrow A_{ij} + \partial_i \partial_j \lambda$$
$$E_{ij} = \partial_0 A_{ij} - \partial_i \partial_j A_0$$
$$B_{[ij]k} = \partial_i A_{jk} - \partial_j A_{ik}$$

The Lagrangian $\mathcal{L} = E^2 - B^2$

These are gapless fractons

- With $SO(3)$, this is the tensor gauge theory of [Xu; Pretko, Senthil; Rasmussen, You, Xu; Pretko; …].
- When J^{ij} is in the $\mathbf{3}'$, A_{ij} is in $\mathbf{3}'$, E_{ij} is in $\mathbf{3}'$, $B_{[ij]k}$ is in $\mathbf{2}$ [Xu, Wu; Ma, Hermele, Chen; Bulmash, Barkeshli; …].

Gauging the dipole symmetry

The pure gauge theory has a global electric $\left(\boldsymbol{R_{time}}, \boldsymbol{R_{space}}\right) = (\mathbf{3}', \mathbf{2})$ tensor symmetry

$$J_0^{ij} \sim E^{ij}$$
$$J^{k(ij)} \sim B^{k(ij)}$$

The equation of motion of A_{ij} leads to the conservation equation and a differential condition follows from Gauss law (the equation of motion of A_0).

The continuum theory (but not the lattice theory) also has a global magnetic $\left(\boldsymbol{R_{time}}, \boldsymbol{R_{space}}\right) = (\mathbf{2}, \mathbf{3}')$ tensor symmetry

$$J_0^{[ij]k} = B^{[ij]k}$$
$$J^{ij} = E^{ij}$$

Let us discuss these tensor symmetries in more general terms.

A tensor global symmetry with the cubic symmetry

The electric symmetry is $\left(\boldsymbol{R_{time}}, \boldsymbol{R_{space}}\right) = (\mathbf{3}', \mathbf{2})$

$$\partial_0 J_0^{ij} = \partial_k J^{k(ij)}$$

In the example, also a differential condition

$$\partial_i \partial_j J_0^{ij} = 0$$

The conserved charges are

$$Q^z = \oint dz\, J_0^{xy} = Q_x^z(x) + Q_y^z(y)$$

and similarly for y, z. The restricted dependence on x, y follows from the differential constraint.

This is a subsystem symmetry.

The lattice version of this system shows that $Q_x^z(x)$, $Q_y^z(y)$ can be discontinuous.

A tensor global symmetry with the cubic symmetry

The magnetic symmetry in the example (not present in the lattice version of this system)

$\left(\boldsymbol{R_{time}}, \boldsymbol{R_{space}}\right) = (\mathbf{2}, \mathbf{3}')$

$$\partial_0 J_0^{[ij]k} = \partial^i J^{jk} - \partial^j J^{ik} \quad i \neq j \neq k$$

The conserved charges are

$$Q^z(z) = \oint dxdy\, J_0^{[xy]z}$$

and similarly for x, y. This is a subsystem symmetry.

Now that we have new global symmetries, we can find new gauge theories. Let us gauge this one.

Gauging the tensor symmetry

Gauge the $(\mathbf{2}, \mathbf{3}')$ tensor symmetry [Slagle, Kim] (suppressing indices and coefficients)

$$\hat{A}_0 J_0 + \hat{A} J$$

$\hat{A}_0$ is in $\mathbf{2}$

$\hat{A}$ is in $\mathbf{3}'$

The gauge parameter $\hat{\lambda}$ is in $\mathbf{2}$

$$\hat{A}_0 \to \hat{A}_0 + \partial_0 \hat{\lambda}$$
$$\hat{A} \to \hat{A} + \partial \hat{\lambda}$$

$\hat{E} = \partial_0 \hat{A} - \partial \hat{A}_0$ is in $\mathbf{3}'$

$\hat{B} = \partial\partial \hat{A}$ is a scalar

The Lagrangian $\mathcal{L} = \hat{E}^2 - \hat{B}^2$

Gauging the tensor symmetry

The pure gauge theory has an electric dipole tensor symmetry with $(\boldsymbol{R_{time}}, \boldsymbol{R_{space}}) = (\mathbf{3}', \mathbf{1})$ (similar to the $(\mathbf{3}, \mathbf{1})$ winding symmetry)

$$\partial_0 J_0^{ij} = \partial^i \partial^j J \quad i \neq j$$
$$\partial^i J_0^{jk} - \partial^j J_0^{ik} = 0$$
$$J_0^{ij} \sim \hat{E}^{ij}$$
$$J \sim \hat{B}$$

The continuum theory (but not the lattice theory) also has a magnetic symmetry, ordinary (scalar) dipole global symmetry

$$\partial_0 J_0 = \partial_i \partial_j J^{ij} \quad i \neq j$$
$$J_0 = \hat{B}$$
$$J^{ij} = \hat{E}^{ij}$$

A dipole tensor global symmetry with the cubic symmetry

The example has a dipole symmetry with $(\boldsymbol{R_{time}}, \boldsymbol{R_{space}}) = (\mathbf{3}', \mathbf{1})$ and a differential condition

$$\partial_0 J_0^{ij} = \partial^i \partial^j J$$

$$\partial^i J_0^{jk} - \partial^j J_0^{ik} = 0$$

The conserved charges are

$$Q^z(z, \mathcal{C}) = \oint_{\mathcal{C} \in (x,y)} \left(dx\, J_0^{xz} + dy\, J_0^{yz} \right)$$

with $\mathcal{C}$ a closed line in the (x, y) plane. (Similarly for Q^x, Q^y.)

The differential condition makes it independent of small deformations of $\mathcal{C}$ (topological in the plane).

The BF-type Chern-Simons theory

Use the A theory to gauge the dipole magnetic symmetry of the $\hat{A}$ theory. (Equivalently, the $\hat{A}$ theory gauges the tensor magnetic symmetry of the A theory.) This leads to [Slagle, Kim]

$$\frac{N}{2\pi}\left(A_0\hat{B} + \hat{A}_0 B + A\partial_0\hat{A}\right) = \frac{N}{2\pi}\left(A_0\partial\partial\hat{A} + \hat{A}_0\partial A + A\partial_0\hat{A}\right)$$

(suppressed indices and coefficients).

It is invariant under

- the cubic group
- the two gauge symmetries
- $\mathbb{Z}_N$ subgroups of the original electric symmetries: a tensor symmetry with $(\mathbf{3}', \mathbf{2})$ and a dipole symmetry with $(\mathbf{3}', \mathbf{1})$

This is the X-cube model.

The $\mathbb{Z}_N$ symmetry operators are its logical operators…

$\mathbb{Z}_N$ tensor symmetry of the X-cube model

Strips generate the $(\mathbf{3}', \mathbf{2})$ symmetry

$$\exp\left(i\int_{z_1}^{z_2} dz \oint_{\mathcal{C}\in(x,y)} \left(A_{zx}dx + A_{zy}dy\right)\right)$$

The dependence on $\mathcal{C}$ in the (x, y) plane is topological.

$\int_{z_1}^{z_2} dz(\dots)$ can include any number of lattice sites.

Special case: $\mathcal{C}$ circles the x direction once and $z_2 = z_1 + a$

$$\exp\left(ia\oint A_{zx}dx\right) = \exp\left(\frac{2\pi i}{N} Q_{xz}^z(z)\right)$$

$Q_{xz}^z(z) \in \mathbb{Z}$ mod N is a discontinuous function of z

This is $\prod Z$ of the X-cube model.

$\mathbb{Z}_N$ dipole symmetry of the X-cube model

Lines generate the $(\mathbf{3}', \mathbf{1})$ dipole symmetry

$$\exp\left(i \oint \hat{A}^{yz} dx \right) = \exp\left(\frac{2\pi i}{N} \left(\hat{Q}_y^{yz}(y) + \hat{Q}_z^{yz}(z) \right) \right)$$

$\hat{Q}_y^{yz}(y)$, $\hat{Q}_z^{yz}(z) \in \mathbb{Z}$ mod N are discontinuous functions of the coordinates.

This is $\prod X$ of the X-cube model.

The fractons, lineon, and planons of the X-cube model are defects in this low energy theory.

These symmetry operators can be interpreted as spacelike versions of these defects.

Summary

- We have studied symmetries with Noether current conservation
$$\partial_0 J_0^I = \partial_i \partial_j \cdots J^{ij\cdots I}$$
J_0^I is in the rotation representation $\boldsymbol{R_{time}}$ and $J^{ij\cdots I}$ is in $\boldsymbol{R_{space}}$. These can be representations of $SO(3)$ or its cubic subgroup.
- Depending on these representations (and the contraction of the indices) the conserved charge is obtained by integrating J_0 over all of space, or over an appropriate subspace $\mathcal{C}$
$$Q(\mathcal{C}) = \oint_{\mathcal{C}} J_0$$
If $\mathcal{C}$ is a proper subspace, this is a subsystem symmetry.

Summary

$$\partial_0 J_0^I = \partial_i \partial_j \cdots J^{ij\cdots I}$$

$$Q(\mathcal{C}) = \oint_{\mathcal{C}} J_0$$

- In addition, J_0^I can be subject to differential conditions. This restricts the dependence of $Q(\mathcal{C})$ on $\mathcal{C}$. For example, it could be topological.
- We gauge the symmetry by writing

$$A_0^I J_0^I + A_{ij\ldots}^I J^{ij\cdots I}$$

The conservation equation leads to a gauge symmetry with λ^I

A_0^I , λ^I in $\boldsymbol{R_{time}}$

$A_{ij\ldots}^I$ in $\boldsymbol{R_{space}}$

A differential condition means that λ^I is itself a gauge field.

Summary

- Such global symmetries and their gauge fields are common in examples, especially in theories of fractons.
- Sometimes these global symmetries are exact symmetries of the short distance problem and sometimes they are emergent global symmetries at long distances.
- The general analysis of the charges is independent of the details of the theory.
- It is important that some of these objects (the fields, the charges, etc.) can be discontinuous functions of the coordinates. This is impossible in a standard continuum quantum field theory.