Dynamics of 2+1d Quantum Field Theories

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IAS

NS, E. Witten, 1602.04251;

NS, T. Senthil, C. Wang, E. Witten, 1606.01989;

P.-S. Hsin, NS, 1607.07457;

O. Aharony, F. Benini, P.-S. Hsin, NS, 1611.07874;

F. Benini, P.-S. Hsin, NS, 1702.07035;

D. Gaiotto, A. Kapustin, Z. Komargodski, NS, 1703.00501;

Z. Komargodski, NS, 1706.08755;

D. Freed, Z. Komargodski, NS, 1707.05448;

D. Gaiotto, Z. Komargodski, NS, 1708.06806;

J. Gomis, Z. Komargodski, NS, 1710.03258;

C. Cordova, P.-S. Hsin, NS, 1711. 10008;

C. Cordova, P.-S. Hsin, NS, 1712.08639.

New elements (will not review here)

- Better understanding of the fermion determinant [Witten] and the role of the spin-charge relation [NS, Witten]
- Boson/fermion duality
  - SUSY dualities in 3d [many papers]
  - Condensed matter motivation [many papers]
  - AdS_4 and large N […; Giombi, Minwalla, Prakash, Trivedi, Wadia, and Yin; …].
  - Finite N version of it [Aharony]
  - Detailed analysis for $N = 1$ [NS, Senthil, Wang, Witten; Karch, Tong]

New elements (will not review here)

- Careful analysis of global symmetries and ‘t Hooft anomalies
  - Gravitational and time-reversal anomalies [many condensed matter papers, Witten]
  - Higher form global symmetries and their anomalies [Kapustin, NS; Gaiotto, Kapustin, NS, Willett; Gaiotto, Kapustin, Komargodski, NS]
  - Global part of the global symmetry group [Benini, Hsin, NS]
  - Many mixed anomalies [many papers]

Steps in analyzing QFT (short distance, kinematics)

- Find the global symmetry (make sure it acts faithfully)
  - Continuous symmetries, e.g. $SU(N_f)$
  - Discrete symmetries, e.g. charge conjugation
  - Time-reversal
  - Supersymmetry
- Find the ‘t Hooft anomalies
  - Global part of the global symmetry group is essential
  - Gravitational anomalies
- …

Steps in analyzing QFT (semiclassical dynamics)

- Look for semiclassical limits, where the analysis is easy
  - Large Chern-Simons coefficient k
  - Large mass m
  - Large N_f
- Relate theories using semiclassical reasoning
  - Give masses to some of the quarks and integrate them out
  - Change the scalar potential

Steps in analyzing QFT (quantum)

- Various limits
  - Large N and holography
- Global symmetry breaking?
- ‘t Hooft anomaly matching conditions
- Dualities
  - including level-rank duality (rigorous)
- Is supersymmetry broken?
- Relate to other theories
  - On domain walls and interfaces of higher dimensional theories (will not discuss here)

Level-rank Duality

Standard in the literature:

- Nk complex fermions lead to

$$\boldsymbol{su}(N)_k \leftrightarrow \boldsymbol{u}(k)_{-N}$$

$$\boldsymbol{u}(N)_{k,k+N} \leftrightarrow \boldsymbol{u}(k)_{-N,-k-N}$$

- Nk real fermions lead to

$$\boldsymbol{so}(N)_k \leftrightarrow \boldsymbol{so}(k)_{-N}$$

- $4Nk$ real fermions lead to

$$\boldsymbol{sp}(N)_k \leftrightarrow \boldsymbol{sp}(k)_{-N}$$

But there are three puzzles…

Level-rank Duality

[Hsin, NS; Aharony, Benini, Hsin, NS; Cordova, Hsin, NS]

Spin:

- Are these TQFTs or spin-TQFTs? Do they depend on the choice of spin structure?
- In many cases spin-TQFT $\leftrightarrow$ TQFT, which cannot be consistent (not even the same number of lines).
- Spins of lines match modulo ½ rather than modulo 1 (even when both sides are non-spin-TQFT).
- Solution: make both sides spin theories – if not spin, tensor an almost trivial TQFT: $\{1, \psi\}$.

Level-rank Duality

[Hsin, NS; Aharony, Benini, Hsin, NS; Cordova, Hsin, NS]

What is the global structure (label the theory by the gauge group of the corresponding Chern-Simons theory)?

Solution:

$$SU(N)_k \leftrightarrow U(k)_{-N}$$

$$U(N)_{k,k+N} \leftrightarrow U(k)_{-N,-k-N}$$

$$Sp(N)_k \leftrightarrow Sp(k)_{-N}$$

$$SO(N)_k \leftrightarrow SO(k)_{-N}$$

E.g. not true for $Spin(N)_k$ (which is usually denoted by $\boldsymbol{so}(N)_k$). See below.

Level-rank Duality

[Hsin, NS; Aharony, Benini, Hsin, NS; Cordova, Hsin, NS]

Couple to background fields:

- metric
- various gauge fields for the global symmetries, e.g. $U(1)$ in $U(N)$, $\boldsymbol{Z}_2 \times \boldsymbol{Z}_2$ $(\mathcal{C}, \mathcal{M})$ in $SO(N)$.
  The duality $\mathcal{L} \leftrightarrow \tilde{\mathcal{L}}$ involves a map of the global symmetries, e.g. $\mathcal{C} \leftrightarrow \mathcal{M}$.

$$\mathcal{L}[A] \leftrightarrow \tilde{\mathcal{L}}[A] + KCS[A]$$

$\mathcal{L}[A]$ is linear in A.

$KCS[A]$ is a Chern-Simons counterterm, which is needed for the duality to be true [Closset, Dumitrescu, Festuccia, Komargodski, NS].

Level-rank Duality

[Hsin, NS; Aharony, Benini, Hsin, NS; Cordova, Hsin, NS]

$$\mathcal{L}[A] \leftrightarrow \tilde{\mathcal{L}}[A] + KCS[A]$$

After doing that, we can add more counterterms to the two sides and gauge the symmetry – make A dynamical.

This leads to new dualities…

Level-rank Duality

[Hsin, NS; Aharony, Benini, Hsin, NS; Cordova, Hsin, NS]

This leads to new dualities

- any one of the three unitary dualities easily leads to the other two and also $U(N)_{k,k-N} \leftrightarrow U(k)_{-N,-N+k}$
- extending the orthogonal dualities, e.g. using $\mathcal{C} \leftrightarrow \mathcal{M}$

$$Spin(N)_k \leftrightarrow O(k)^0{}_{-N,-N}$$

$$O(N)^1{}_{k,k-1} \leftrightarrow O(k)^1{}_{-N,-N+1}$$

in $O(N)^r_{k,L}$ the level k is the ordinary Chern-Simons term and $r\,\pi\, w_2 \cup w_1 + L\, f(w_1)$ are discrete θ-terms with r an integer modulo 2 and L an integer modulo 8. $f(w_1)$ an allowed “local” term.

Level-rank Duality

[Hsin, NS; Aharony, Benini, Hsin, NS; Cordova, Hsin, NS]

Compare with known dualities with $\mathcal{N} = 2$ SUSY

- $SU(N)_k \to (SU(N)_{k+N})^S$
- $U(N)_{k,k'} \to \left(U(N)_{k+N,k'}\right)^S$
- $SO(N)_k \to \left(SO(N)_{k+N-2}\right)^S$
- $Spin(N)_k \to \left(Spin(N)_{k+N-2}\right)^S$
- $O(N)^r_{k,L} \to \left(O(N)^{r+1}_{k+N-2,L+N-1}\right)^S$

and opposite shift for negative k.

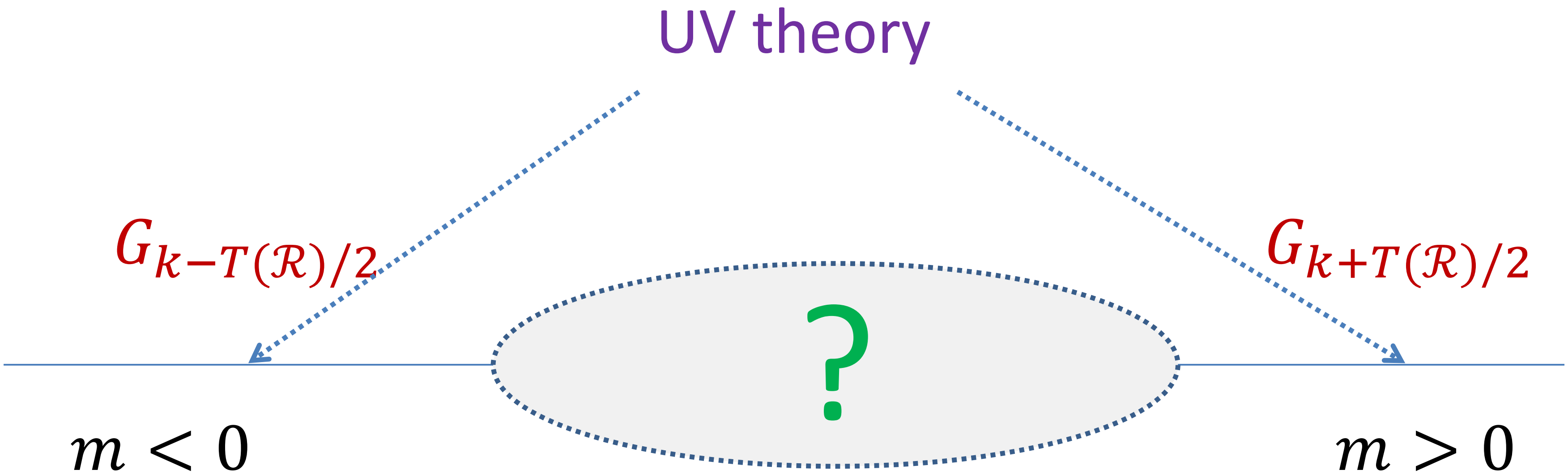
With these shifts it agrees with [Giveon, Kutasov; Kapustin; Aharony, Razamat, NS, Willett] ($f(w_1)$ was ignored there).

The theories

- Gauge group G
  - $U(1)$
  - $SU(N)$
  - $SO(N), Spin(N)$
  - $Sp(N)$
- Fermions in a representation $\mathcal{R}$
  - N_f fundamentals
  - Adjoint
  - Symmetric or anti-symmetric tensors
- Chern-Simons terms (and discrete θ-parameters)

G_k with fermions in a representation $\mathcal{R}$
Large mass $|m|$

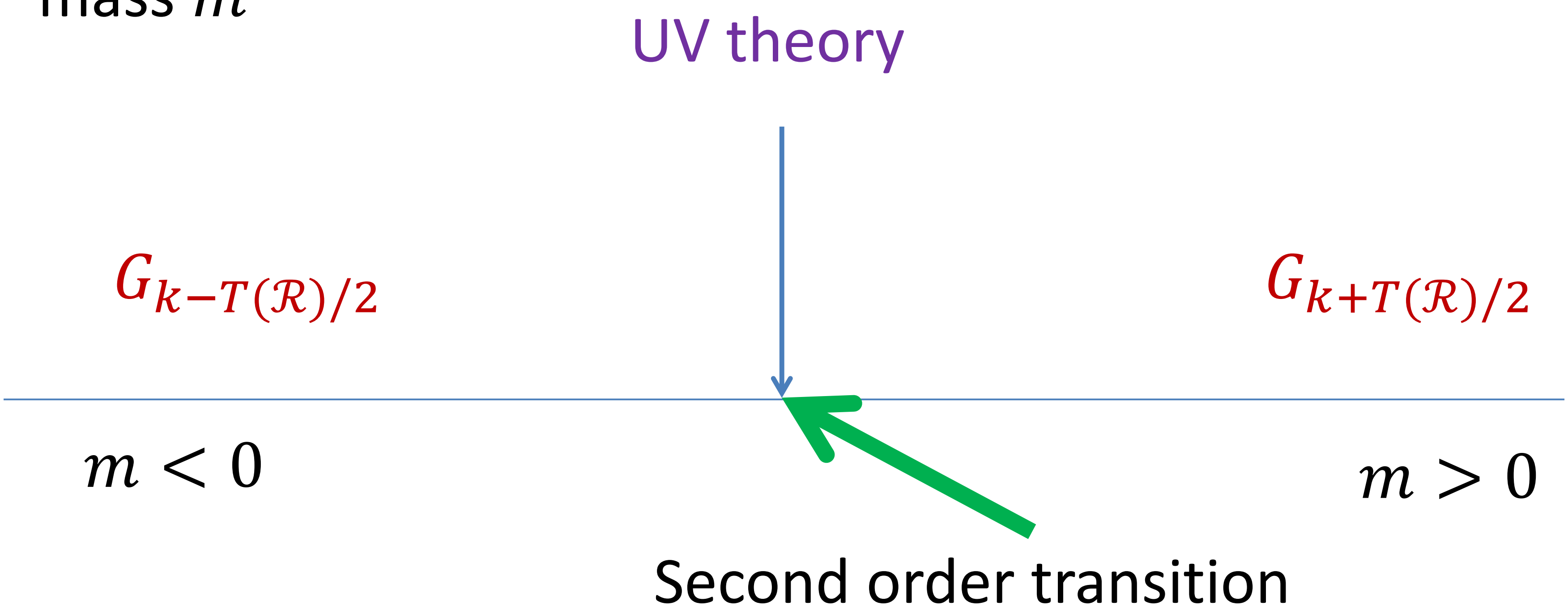
- At low energy a TQFT $G_{k\pm T(\mathcal{R})/2}$. Need k=($T(\mathcal{R})/2$)mod 1, where $T(\mathcal{R})$ is the index of $\mathcal{R}$.



  – What happens at small mass?

G_k with fermions in a representation $\mathcal{R}$
Large level k

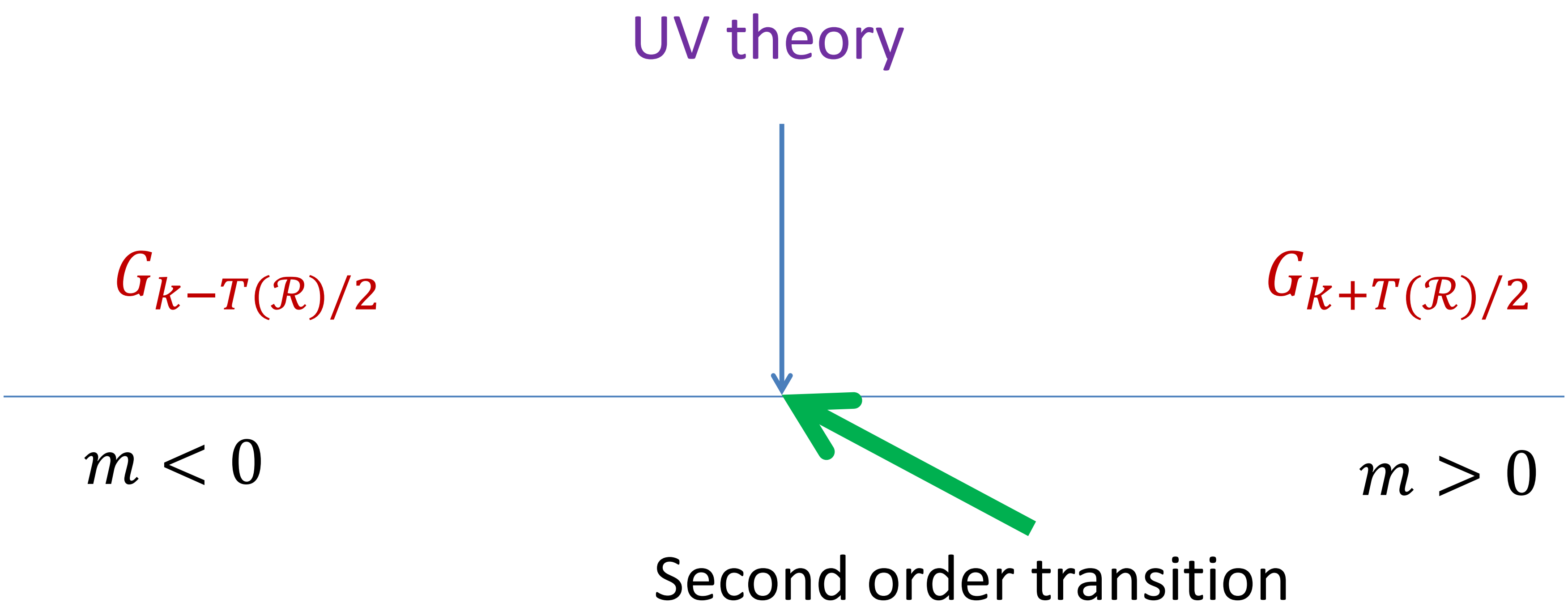
- Essentially free fermions with a Gauss law constraint
- Second order transition as a function of the fermion mass m



– What happens at small k?

G_k with fermions in a representation $\mathcal{R}$
Large $T(\mathcal{R})$ (e.g. large N_f)

- Second order transition as a function of the fermion mass m [Appelquist, Nash]

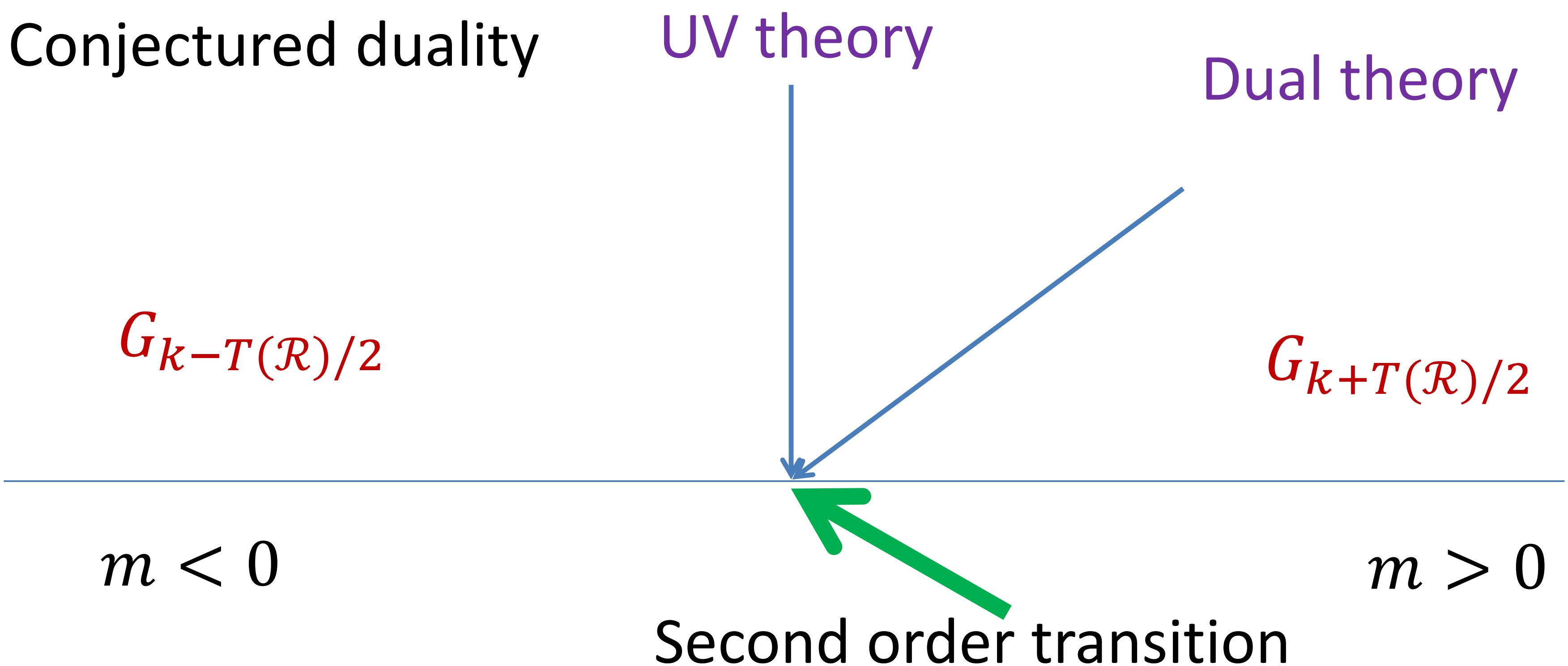


  – What happens at small N_f?

CONJECTURES
AHEAD

G_k with fermions in a representation $\mathcal{R}$
Large k

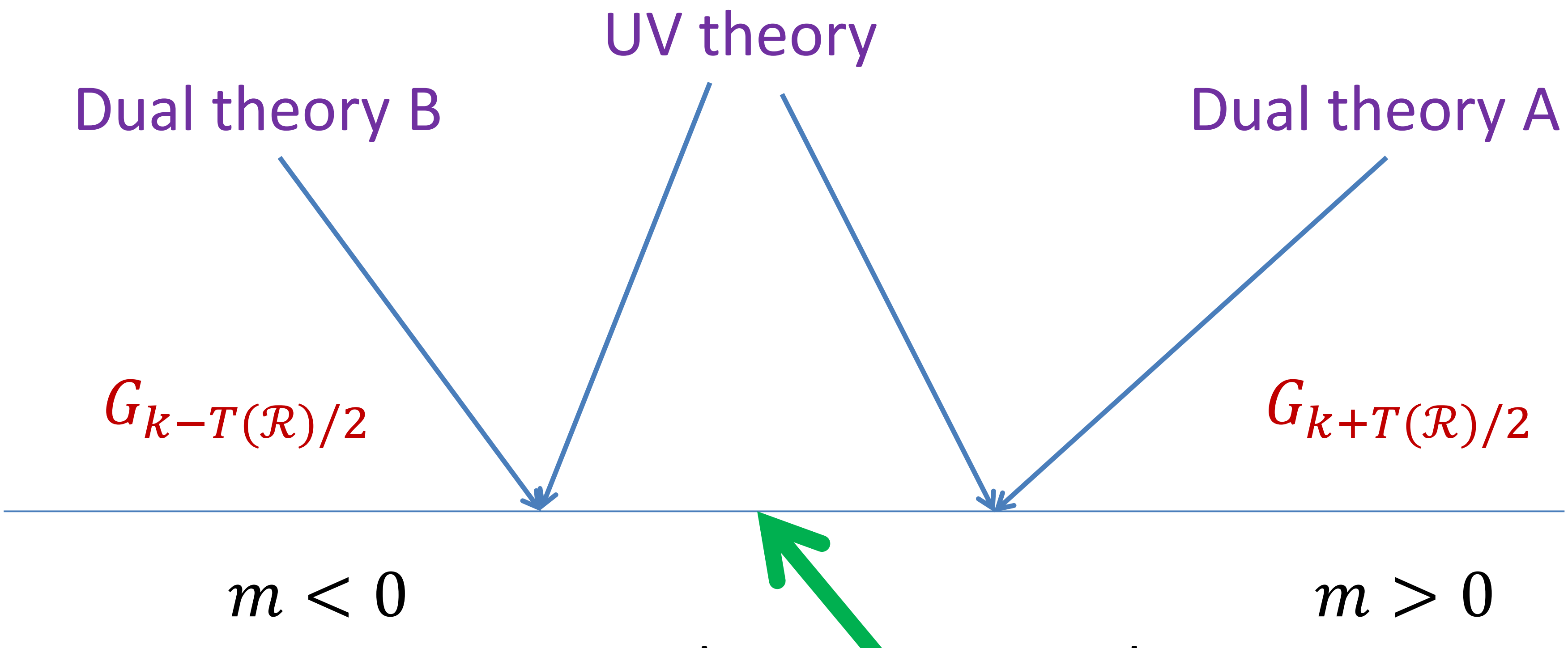
Conjectured duality



In the dual theory description $G_{k\pm T(\mathcal{R})/2}$ are replaced by dual TQFTs (e.g. level-rank duality).

G_k with fermions in a representation $\mathcal{R}$
Small level k, small mass m, and small N_f

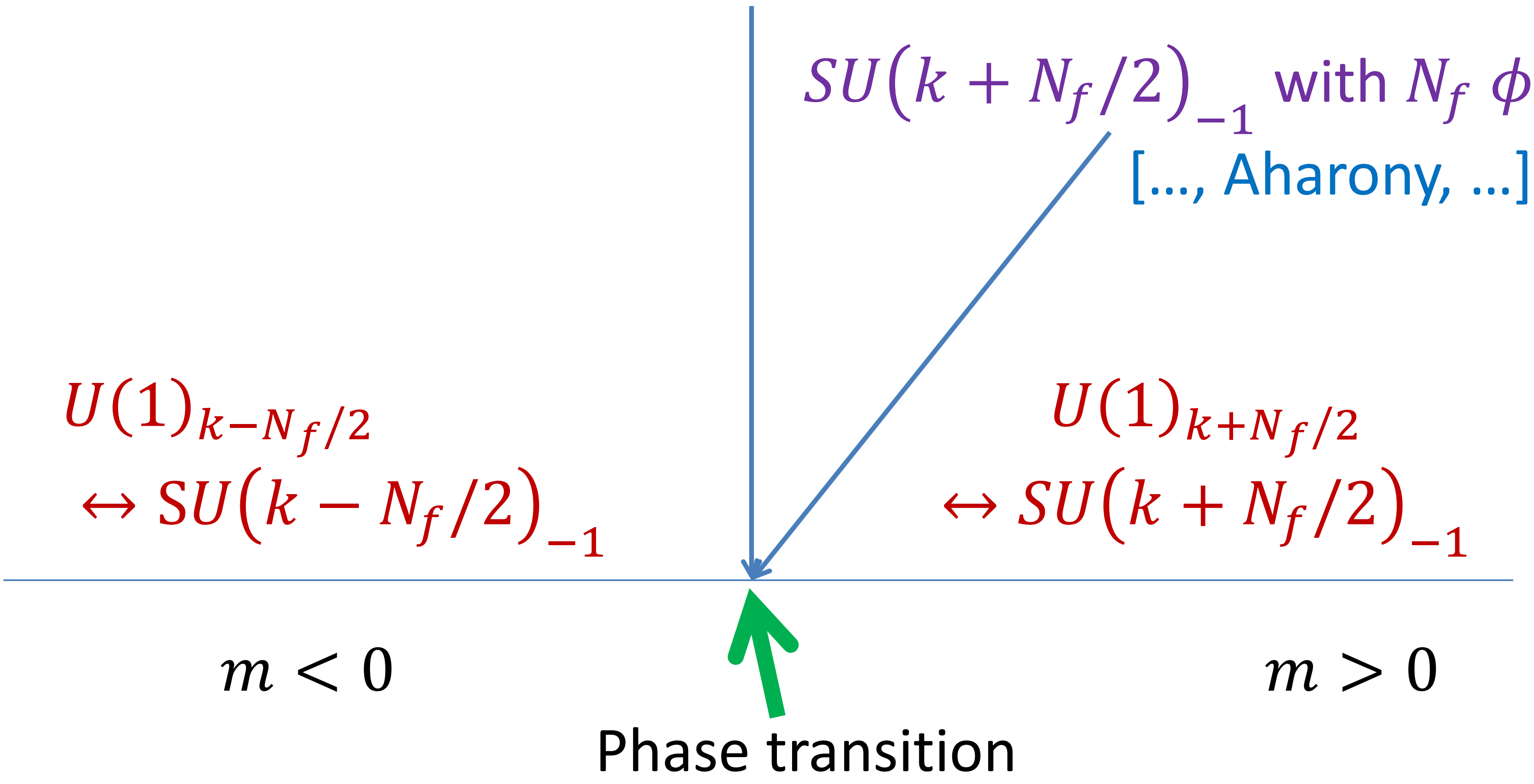
[Komargodski, NS]



Intermediate quantum phase.
It and the transitions are described in
the dual theories via weak coupling.

$U(1)_k$ with N_f electrons

$$\left(k = \frac{N_f}{2} \bmod 1\right) \text{ with } k > \frac{N_f}{2}$$



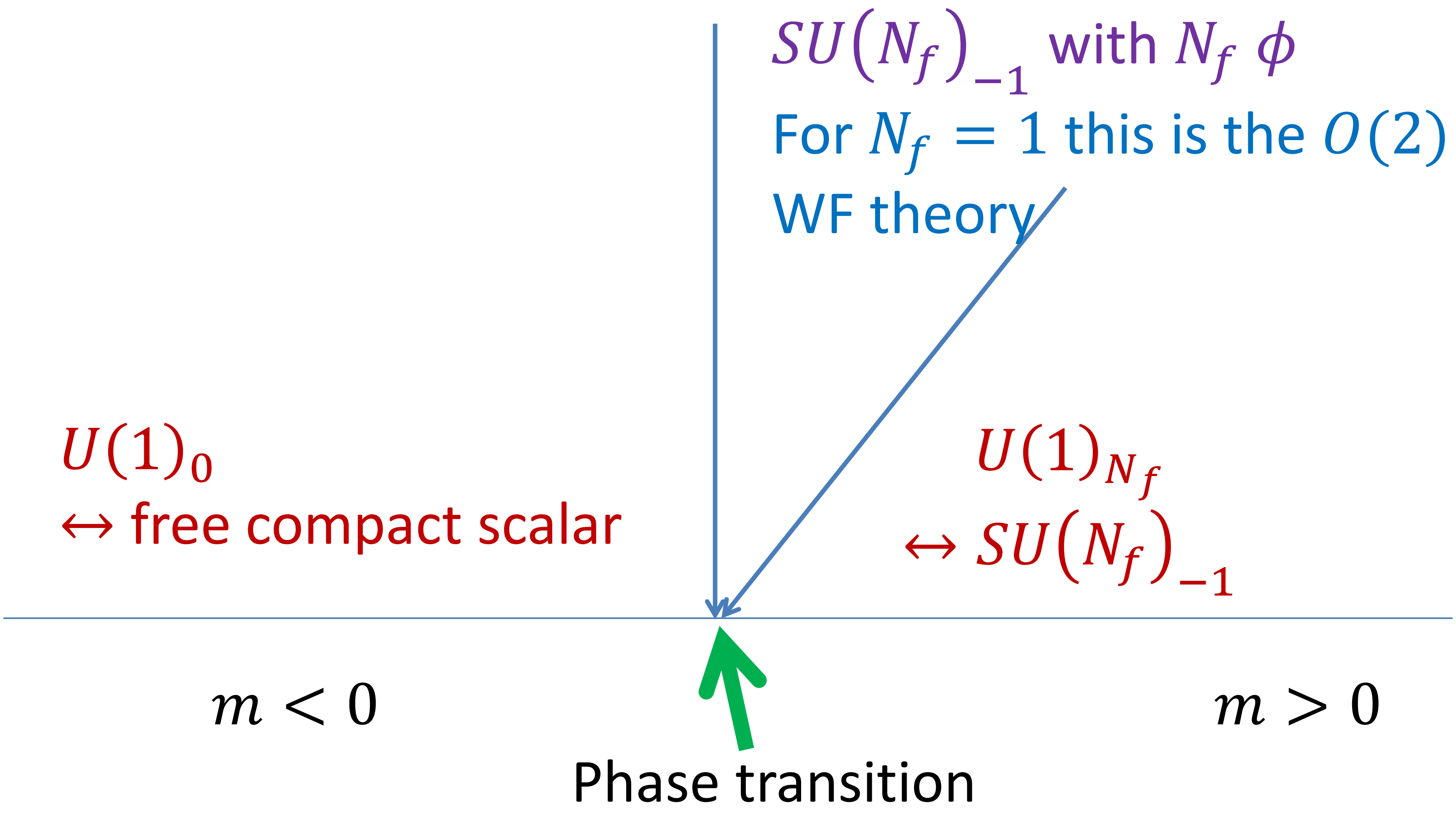
$U(1)_k$ with N_f electrons

- Did not prove it
- Did not determine whether the transitions are first or second order
- At large k the transition is second order. Conjecture that it is second order for all k.
- The duality originated from another duality at large N and k with $N_f = 1$ by Giombi, Minwalla, Prakash, Trivedi, Wadia, and Yin (we have $N = 1$)
- This statement for generic N, k, and N_f was found by Aharony and was made more precise in [Hsin, NS]

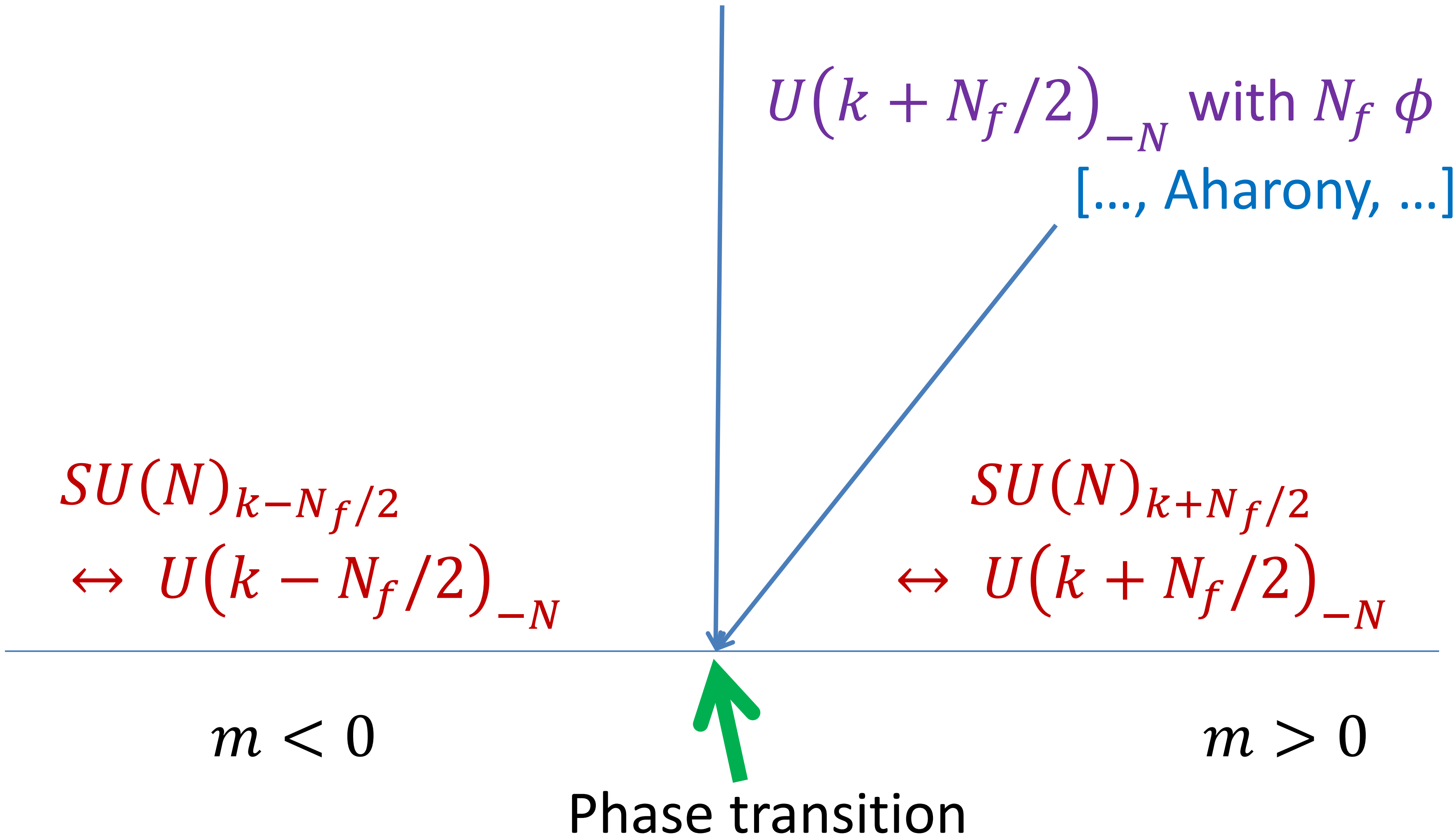
$U(1)_k$ with N_f electrons

- In the scalar picture there is a $|\phi|^4$ interaction and the duality is valid in the IR (gauged Wilson Fisher fixed point).
- In the scalar picture the gauge group is Higgsed and the level does not change.
- Monopoles $\leftrightarrow$ baryons $= \phi^{k+N_f/2}$.
- The dual description reproduces the same gapped phases using level-rank duality

$U(1)_{k=N_f/2}$ with N_f electrons



$SU(N)_k$ with N_f quarks $N_f/2 \le k$



$SU(N)_k$ with N_f quarks

$2k < N_f < N^*(N,k)$ [Komargodski, NS]

$U(N_f/2-k)_N$ with N_f ϕ

$U(N_f/2+k)_{-N}$ with N_f ϕ

$SU(N)_{k-N_f/2}$

$\leftrightarrow U(N_f/2-k)_N$

$SU(N)_{k+N_f/2}$

$\leftrightarrow U(N_f/2+k)_{-N}$

$m<0$

$$\frac{U(N_f)}{U\left(\frac{N_f}{2}+k\right)\times U\left(\frac{N_f}{2}-k\right)}$$

with $N\Gamma_{\mathrm{WZ}}$

$m>0$

$SU(N)_k$ with N_f quarks
$2k < N_f < N^*(N,k)$

- Three phases
  - For large $|m|$ semiclassical physics – gapped, topological.
  - For small $|m|$ a new quantum phase with global symmetry breaking $U(N_f)/U\left(\frac{N_f}{2}+k\right) \times U\left(\frac{N_f}{2}-k\right)$
  - Each phase transition has a weakly coupled bosonic dual description
- The intermediate phase
  - Wess-Zumino term from the Chern-Simons term
  - For $k=0$: $U(N_f) \to U(N_f/2) \times U(N_f/2)$ [Vafa, Witten] with a WZ term
  - Assuming this, we can derive for other values of k
  - Skyrmions in the nonlinear model are monopoles in the bosonic theory and are the baryons in the fermionic theory

$SO(N)_k$ with N_f quarks $\frac{N_f}{2} \leq k$

$SO\left(k + N_f/2\right)_{-N}$ with N_f ϕ

[Aharony, Benini, Hsin, NS]

$SO(N)_{k-N_f/2}$

$\leftrightarrow SO\left(k - N_f/2\right)_{-N}$

$SO(N)_{k+N_f/2}$

$\leftrightarrow SO\left(k + N_f/2\right)_{-N}$

$m < 0$

$m > 0$

Phase transition

$SO(N)_k$ with N_f quarks

$2k < N_f < N^*(N,k)$ [Komargodski, NS]

$SO\left(N_f/2 - k\right)_N$ with N_f ϕ

$SO\left(N_f/2 + k\right)_{-N}$ with N_f ϕ

$SO(N)_{k-N_f/2}$

$\leftrightarrow SO\left(N_f/2 - k\right)_N$

$SO(N)_{k+N_f/2}$

$\leftrightarrow \ SO\left(N_f/2 + k\right)_{-N}$

$m < 0$

$$\frac{SO(N_f)}{SO\left(\frac{N_f}{2}+k\right)\times SO\left(\frac{N_f}{2}-k\right)}$$

with $N\Gamma_{\mathrm{WZ}}$

$m > 0$

$SO(N)_k$ with N_f quarks
$2k < N_f < N^*(N,k)$

- As in $SU(N)$, three phases
- The intermediate phase
  - Wess-Zumino term from the Chern-Simons term
  - $\boldsymbol{Z}_2$ magnetic symmetry in the UV
    - maps to a $\boldsymbol{Z}_2$charge conjugation in the dual bosonic theories
    - It acts on the sigma model – it is spontaneously broken – monopole condensation
  - $\boldsymbol{Z}_2$ charge conjugation symmetry in the UV
    - maps to a $\boldsymbol{Z}_2$magnetic symmetry in dual bosonic theories
    - $\boldsymbol{Z}_2$ baryons in the UV are $\boldsymbol{Z}_2$monopoles in the bosonic duals. Skyrmions in the nonlinear model

$Spin(N)_k$ with N_f quarks

$2k < N_f < N^*(N,k)$

[Komargodski, NS; Cordova, Hsin, NS]

$O\left(\frac{N_f}{2} - k\right)^0_{N,N}$ with N_f ϕ

$O\left(\frac{N_f}{2} + k\right)^0_{-N,-N}$ with N_f ϕ

$Spin(N)_{k-N_f/2}$

$\leftrightarrow O\left(\frac{N_f}{2} - k\right)^0_{N,N}$

$Spin(N)_{k+N_f/2} \leftrightarrow$

$O\left(\frac{N_f}{2} + k\right)^0_{-N,-N}$

$m < 0$

$$\frac{O(N_f)}{O\left(\frac{N_f}{2}+k\right)\times O\left(\frac{N_f}{2}-k\right)}$$

with $N\Gamma_{\text{WZ}}$

$m > 0$

$Spin(N)_k$ with N_f quarks $2k < N_f < N^*(N,k)$

- Comparing with the $SO(N)$ theory we gauged the $\boldsymbol{Z}_2$ magnetic symmetry.
- This maps under the duality to gauging the $\boldsymbol{Z}_2$ charge conjugation in the dual bosonic theories (the other levels follow from the map of the counterterms).
- Gauging this $\boldsymbol{Z}_2$ symmetry in the sigma model amounts to a quotient by the global $\boldsymbol{Z}_2$.
- This leads to a nontrivial π_1 and to strings – confining strings. (Similar confining strings in the analogous 4d problem [Witten].)

$Spin(N)_k$ with N_f quarks
$2k < N_f < N^*(N,k)$

- Equivalently, the UV $Spin(N)$ theory has a global $\boldsymbol{Z}_2$ one-form symmetry associated with its center. It is realized in the IR as the nontrivial π_1.

The consistency of this picture is a nontrivial test of our proposal.

More (will not discuss here)

- ${O(N)^r}_{k,L}$ with fermions in the fundamental representation
- $Sp(N)_k$ with fermions in the fundamental representation
- Relation to theories on domain walls and interfaces in four dimensions

$SU(N)_k$ with λ in adjoint for $N \leq 2k$

SUSY

$SU(N)_{k-\frac{N}{2}} \leftrightarrow$
$U(k - N/2)_{-N}$

$SU(N)_{k+\frac{N}{2}} \leftrightarrow$
$U(k + N/2)_{-N}$

$m < 0$

$m > 0$

Phase transition

No transition. Supersymmetry is unbroken [Witten].

Holographic duals [Maldacena, Nastase] with $k - \frac{N}{2}$ branes.

$SU(N)_k$ with λ in adjoint for $0 \le 2k < N$

[Gomis, Komargodski, NS]

$U\left(\frac{N}{2} - k\right)_{\frac{3N}{4}+\frac{k}{2},N}$ with $\hat{\lambda}$

SUSY

$U\left(\frac{N}{2} + k\right)_{-\frac{3N}{4}+\frac{k}{2},-N}$ with $\tilde{\lambda}$

$SU(N)_{k-\frac{N}{2}} \leftrightarrow$
$U(N/2 - k)_N$

$SU(N)_{k+\frac{N}{2}} \leftrightarrow$
$U(k + N/2)_{-N}$

$m < 0$

$m > 0$

$U(N/2 + k)_{-N/2+k,-N}$
$\leftrightarrow \; U(N/2 - k)_{N/2+k,N}$

Massless Goldstino [Witten].

Phase transitions

$SU(N)_k$ with λ in adjoint for $0 \le 2k < N$

- Three phases
  - For large $|m|$ semiclassical physics – gapped, topological.
  - For small $|m|$ a new quantum phase
$$U\left(\frac{N}{2}+k\right)_{-\frac{N}{2}+k,-N} \leftrightarrow U\left(\frac{N}{2}-k\right)_{\frac{N}{2}+k,N}$$
  - Each phase transition has a fermionic dual description.
  - $m_\lambda = 0$ supersymmetry. Spontaneously broken with a massless Goldstino [Witten] in addition to the TQFT.
- For special N and k the picture simplifies.
- Many consistency checks

More (will not discuss here)

- $SO(N)_k, Spin(N)_k, O(N)^r_{k,L}$ with adjoint fermions. (Highly nontrivial consistency tests of our picture.)
- $Sp(N)_k$ with adjoint fermions
- $SO(N)_k$ with fermions in a traceless symmetric tensor
- $Sp(N)_k$ with fermions in a traceless anti-symmetric tensor

They exhibit new phenomena.

Fermion-Fermion Dualities

[NS, T. Senthil, Wang, Witten]

Starting with the boson-fermion dualities (with the necessary background fields), we can derive a fermion-fermion duality (earlier version by [Son; Wang, Senthil; Metlitski, Vishwanath])

$$i\bar{\Psi} \not{D}_A \Psi \quad \leftrightarrow$$

$$i\bar{\chi} \not{D}_a \chi - \frac{2}{4\pi} bdb + \frac{1}{2\pi} adb + \frac{1}{2\pi} Adb - \frac{1}{4\pi} AdA$$

A is a classical $U(1)$ background field. (Note its counterterm.)

The dynamics in the RHS is not that of QED (different critical exponents). It leads in the IR to the free LHS.

Fermion-Fermion Dualities [Cordova, Hsin, NS]

Using this duality we can derive

$$U(1)_0 + \psi \text{ with charge } 2 \leftrightarrow \chi + U(1)_2$$

- Both sides are time-reversal invariant at all scales
  - unlike $U(1)_{1/2}$ with a single fermion of charge one
- The RHS includes a free Dirac fermion and a decoupled TQFT. Can view the LHS as UV and the RHS as IR.
- The monopole operator in the UV is mapped to the free fermion χ in the IR.
- The Wilson line in the LHS is the Wilson line of $U(1)_2$.

Fermion-Fermion Dualities [Cordova, Hsin, NS]

$$U(1)_0 + \psi \text{ with charge } 2 \leftrightarrow \chi + U(1)_2$$

- The magnetic global $U(1)_M$ acts in the RHS both on χ and on the anyons of the TQFT.
- All local operators satisfy $(-1)^M = (-1)^F$.
- As is common in dualities (using standard definitions),

$$T_{UV} \leftrightarrow (CT)_{IR}$$
$$(CT)_{UV} \leftrightarrow T_{IR}$$

- The anomalies in T and CT match
  - $\nu_{T_{UV}} = 0, \ \nu_{(CT)_{UV}} = 2$
  - $\nu_{T_{IR}} = 2, \ \nu_{(CT)_{IR}} = 0$

Fermion-Fermion Dualities [Cordova, Hsin, NS]

$$U(1)_0+\psi\ \text{with charge}\ 2\leftrightarrow \chi+U(1)_2$$

- Add to the UV Lagrangian a charge 2 monopole operator.
  - Cannot add a charge 1 monopole operator because it is a fermion.
- In the IR it is $\chi_1^2-\chi_2^2$ with $\chi=\chi_1+i\chi_2$. It splits the IR fixed point to two points with massless $\chi_{1,2}$.
- It breaks the magnetic $U(1)_M$ to $\boldsymbol{Z}_2$: $(-1)^M=(-1)^F$.
- It preserves C and breaks T_{UV}.
- But…

Fermion-Fermion Dualities [Cordova, Hsin, NS]

$U(1)_0 + \psi$ with charge $2 \leftrightarrow \chi + U(1)_2$

- It preserves a subgroup – modified time-reversal

$$T' = T_{UV} e^{\frac{i\pi}{2}M} = (CT)_{IR} e^{\frac{i\pi}{2}M}$$

- Its algebra is not the standard one

$$T'C = CT'(-1)^M$$
$$(CT')^2 = (-1)^M = (-1)^F$$
$$(T')^2 = (-1)^F(-1)^M = 1$$

T' is not a conventional time-reversal symmetry. CT' is.

Such a deformation of the naïve global symmetry group is ubiquitous.

More

- Many other cases have been analyzed.
  - They exhibit many new phenomena.
- Many more generalizations have not yet been analyzed.

QFT is fun