Duality in 2 + 1 Dimensions

Nathan Seiberg

IAS

NS and E. Witten, arXiv:1602.04251;

NS, T. Senthil, C. Wang, and E. Witten, arXiv:1606.01989;

P.-S. Hsin, NS, arXiv:1607.07457

O. Aharony, F. Benini, P.-S. Hsin, NS, arXiv:1611.07874

F. Benini, P.-S. Hsin, NS, arXiv:1702.07035

Three (almost) independent lines of development – the unity of physics

- The condensed matter, $3d$ quantum field theory route
- The supersymmetric route
- The AdS/CFT, large N route

The condensed matter, $3d$ QFT route

- Statistical transmutation: a massive particle coupled to a dynamical (statistical, emergent) gauge field with a Chern-Simons term can change its spin and statistics [Wilczek; Polyakov; Zhang, Hansson, Kivelson ; Jain; …].
  - Many applications (FQHE, composite fermions, flux attachment, …)

- This does not mean that a second-quantized theory of massless interacting bosons coupled to a gauge field with a Chern-Simons term is dual to a theory of fermions (or the other way around).

The condensed matter, $3d$ QFT route

Particle/vortex duality [Peskin; Dasgupta and Halperin]

$$|D_B\Phi|^2 - |\Phi|^4 \quad \leftrightarrow \quad |D_b\hat{\Phi}|^2 - |\hat{\Phi}|^4 + \frac{1}{2\pi}Bdb$$

- LHS is XY, $O(2)$ Wilson-Fisher.
- B is a background field coupled to a global $U(1)_B$ symmetry.
- RHS is a gauged version of this theory. b is a dynamical field.
- IR duality – two different theories flowing to the same IR fixed point.
- $\Phi \leftrightarrow \mathcal{M}_\mathrm{b}$ is a monopole operator of b (charged under $U(1)_B$).
- $|\Phi|^2 \leftrightarrow -\left|\widehat{\Phi}\right|^2$. Upon deformation: unbroken $U(1)_B$ phase is Higgs phase in the RHS; broken $U(1)_B$ phase massless b.

The condensed matter, $3d$ QFT route

Boson/fermion duality [Chen, Fisher, Wu; Barkeshli, McGreevy; … Karch, Tong]

$$|D_B\Phi|^2 - |\Phi|^4 \quad \leftrightarrow \quad i\bar{\Psi}\not{D}_a\Psi + \frac{1}{2\pi}Bda$$

LHS Wilson-Fisher fixed point (B is a background gauge field)

RHS QED with gauge field a with a single fermion, a.k.a $U(1)_{1/2}$

- Arguments involve elementary fields with fractional charges and fractional level Chern-Simons terms.
- LHS is T-reversal invariant, while RHS seems like it is not.
- LHS does not need a spin structure, while RHS does. Violating gravitational ‘t Hooft matching conditions?
- The IR behavior of the RHS is debated.

The condensed matter, $3d$ QFT route

Fermion/fermion duality [Son; Wang, Senthil; Metlitski, Vishwanath]

$$i\bar{\Psi}\not{D}_A\Psi \quad \leftrightarrow \quad i\bar{\chi}\not{D}_a\chi + \frac{1}{4\pi}Ada$$

- Motivated by
  - physics of the lowest Landau level at half-filling [Halperin, Lee, Read]
  - T-Pfaffian state of topological insulators [Chen, Fidkowski, Vishwanath].
- Improperly quantized Chern-Simons term
- LHS is T-reversal invariant (with anomaly) and RHS seems like it is not. Its IR behavior is debated.

The supersymmetric route

- Many dualities of $4d$ $\mathcal{N}=1$ theories (IR dualities) [NS; …]
- They motivated many dualities in $3d$
  - $\mathcal{N}=2$ [Aharony, Hanany, Intriligator, NS, Strassler; Aharony; Giveon, Kutasov; … Benini, Closset, Cremonesi; Intriligator, NS; Aharony, Razamat, NS, Willett; Park, Park; …]
  - $\mathcal{N}=4$ $3d$ mirror symmetry [Intriligator, NS; …]
- These use particle/vortex duality
- Later derived by compactification of $4d$ $\mathcal{N}=1$ dualities on a circle and then flow with relevant operators [Aharony, Razamat, NS, Willett].
  - More checks
  - Leads to many new dualities

The supersymmetric route

- Many checks using supersymmetry and localization
- Related to string duality
- Connected to level/rank duality of $3d$ topological quantum field theory and $2d$ RCFT (rigorous [...; Hsin, NS]).

- Can flow from them to non-supersymmetric theories [Jain, Minwalla, Yokoyama; Gur-Ari, Yacoby; ... ; Kachru, Mulligan, Torroba, Wang].
  - This motivates non-supersymmetric dualities.
  - But the flow might not be smooth.

The AdS/CFT, large N route

- Same $4d$ Vasiliev theory is dual to two different $3d$ field theories [Vasiliev ; Sezgin, Sundell; Klebanov, Polyakov; Giombi, Yin; Aharony, Gur-Ari, Yacoby; Giombi, Minwalla, Prakash, Trivedi, Wadia, Yin].
  - Scalars coupled to a Chern-Simons gauge theory
  - Fermions coupled to a Chern-Simons gauge theory
- Hence, a purely field theoretic duality between them
- Many explicit checks of this duality at large N [Maldacena, Zhiboedov; Aharony, Giombi, Gur-Ari, Maldacena, Yacoby; Jain, Minwalla, Sharma, Takimi, Wadia, Yokoyama; Minwalla, Yokoyama; Yokoyama; Jain, Mandlik, Minwalla, Takimi, Wadia, Yokoyama; Inbasekar, Jain, Mazumdar, Minwalla, Umesh, Yokoyama; …]

Synthesis [Aharony]

3 [Aharony] (+ 1 [Hsin, NS])

N_f scalars at $|\Phi|^4$ point coupled to $\quad\quad$ N_f fermions coupled to

- $SU(N)_k \quad \leftrightarrow \quad U(k)_{-N+N_f/2,-N+N_f/2}$
- $U(N)_{k,k} \quad \leftrightarrow \quad SU(k)_{-N+N_f/2}$
- $U(N)_{k,k+N} \quad \leftrightarrow \quad U(k)_{-N+N_f/2,-N-k+N_f/2}$
- $U(N)_{k,k-N} \quad \leftrightarrow \quad U(k)_{-N+N_f/2,-N+k+N_f/2}$

Fits the large N picture ($N, k \to \infty$ with finite N/k)

Fits the supersymmetric picture

Consistent with mass deformations

Related to level/rank duality (set $N_f = 0$)

Baryon and monopole operators match [Radicevic]

Additional conjectured dualities

[Aharony, Benini, Hsin, NS]

N_f scalars at $\|\Phi\|^4$ point coupled to		N_f fermions coupled to
• $SO(N)_k$	$\leftrightarrow$	$SO(k)_{-N+N_f/2}$
• $SP(N)_k$	$\leftrightarrow$	$SP(k)_{-N+N_f/2}$

Fits the large N picture ($N, k \to \infty$ with finite N/k)

Difference between $SO(N)$ and $SP(N)$ in higher orders in $1/N$

Fits the supersymmetric picture

Consistent with mass deformations

Related to level/rank duality (set $N_f = 0$)

Can gauge charge conjugation or monopole number to turn into $O(N)$ or $Spin(N)$ or $Pin^{\pm}(N)$

Puzzles

- Is it true for all N, k, N_f?
- How can a theory of bosons, which does not need a spin structure be dual to a theory of fermions, which needs it?
- What is the relation to the dualities in the condensed matter literature (with puzzles about quantization of coefficients, T-reversal invariance, etc.)?
- What is the precise statement of the dualities (including the coupling to background gauge fields and their Chern-Simons counterterms)?
- Are the assumptions independent? Can we assume some of these dualities and derive others?
- Are there other such dualities?

Digression: The Parity Anomaly

Consider a single complex fermion coupled to a gauge field A.

Its partition function is [Alvarez-Gaumé, Della Pietra, Moore]

$$Z = \left|\mathrm{Det}\, \not{D}_A\right| \exp\left(\pm \frac{i\pi\eta(A)}{2}\right)$$

$$\eta(A) = \lim_{\epsilon\to 0^+} \sum_i \exp(-\epsilon|\lambda_i|)\,\mathrm{sign}\ (\lambda_i)$$

The sign ambiguity reflects the T-breaking.

Cannot write the phase as $\exp\left(\pm\frac{i}{8\pi}\int AdA\right)$, because this Chern-Simons term is not well defined

$$\exp\left(\pm\frac{i\pi\eta}{2}\right) = \pm\exp\left(\pm\frac{i}{8\pi}\int AdA + \text{gravitational term}\right)$$

Digression: The Parity Anomaly

Note, the standard lore states:

"The free massless fermion theory is not gauge invariant and therefore we have to add an improperly quantized Chern-Simons counterterm $\frac{1}{8\pi} AdA +$ *gravitational terms."*

Instead, the massless fermion theory is gauge invariant and no counterterm has to be added.

We will choose the minus sign $Z = |\mathrm{Det} \not{D}_A| \exp\left(-\frac{i\pi\eta(A)}{2}\right)$

Then for a massive fermion

$$Z = |Z| \qquad \text{for} \quad m > 0$$

$$Z = |Z| \exp\left(-\frac{i}{4\pi}\int AdA + \text{gravitational terms}\right) \qquad \text{for} \quad m < 0$$

Digression: Spin/Charge Relation

Consider a system of electrons with an arbitrary Hamiltonian such that the nuclear spins are not important.

All the states of the system (in finite volume) and all the local operators must satisfy a selection rule:

$$2 \text{ Spin} = U(1)_A \text{ Charge mod } 2$$

This is a powerful constraint on any long distance description.

One way of thinking about the spin/charge constraint is similar to ‘t Hooft anomaly constraints…

Digression: Spin/Charge Relation

Following ‘t Hooft, we couple the microscopic system to a background metric and a background A.

Naively, since we have spinors, we need the background manifold to have a spin structure.

But, we can also place the system on a non-spin manifold by letting A be a $Spin_c$ connection...

Digression: Spin/Charge Relation

We can also place the system on a non-spin manifold by letting A be a $Spin_c$ connection.

(This means that the obstruction to spin structure is corrected by A not being a $U(1)$ gauge field; i.e. $\int \frac{F}{2\pi} = \frac{1}{2} \int w_2 \bmod \mathbf{Z}$.)

Note, we are not really interested in the behavior of our system of electrons in this more general background. This is merely a device to constrain the long distance behavior .

Therefore, the macroscopic theory should also satisfy this selection rule.

Digression: Spin/Charge Relation

The same argument can be repeated for two different dual theories.

We can have two dual theories such that

- Neither theory needs a spin manifold and a choice of spin structure (e.g. bosonic particle/vortex duality)
- Both theories need a spin structure. (This is common in supersymmetric theories.)
- One theory needs a spin structure and the other does not, but both theories can be coupled to a background $Spin_c$ connection. Then, both theories can be formulated on a non-spin manifold.

A new boson/boson duality

Start with the standard particle/vortex duality

$$|D_B\Phi|^2 - |\Phi|^4 \quad \leftrightarrow \quad |D_{\hat{b}}\hat{\Phi}|^2 - |\hat{\Phi}|^4 + \frac{1}{2\pi}Bd\hat{b}$$

Derive other dualities by changing the two sides:

- Add a Chern-Simons counterterm for the classical field B
- Gauge it by turning B into a dynamical field b and adding a new classical gauge field A.
- Use other dualities.
- Repeat.

A new boson/boson duality

$$|D_B\Phi|^2 - |\Phi|^4 \quad \leftrightarrow \quad |D_{\hat{b}}\hat{\Phi}|^2 - |\hat{\Phi}|^4 + \frac{1}{2\pi}Bd\hat{b}$$

Add to the two sides $\frac{1}{4\pi}BdB + \frac{1}{2\pi}AdB$ and turn B into a dynamical $U(1)$ gauge field b

$$|D_b\Phi|^2 - |\Phi|^4 + \frac{1}{4\pi}bdb + \frac{1}{2\pi}Adb \quad \leftrightarrow$$

$$|D_{\hat{b}}\hat{\Phi}|^2 - |\hat{\Phi}|^4 + \frac{1}{2\pi}bd\hat{b} + \frac{1}{4\pi}bdb + \frac{1}{2\pi}Adb$$

In the right hand side b represents an almost trivial theory $U(1)_1$ and it can be integrated out…

A new boson/boson duality

$$|D_b\Phi|^2 - |\Phi|^4 + \frac{1}{4\pi} bdb + \frac{1}{2\pi} Adb \quad \leftrightarrow$$

$$|D_{\hat{b}}\hat{\Phi}|^2 - |\hat{\Phi}|^4 - \frac{1}{4\pi}\hat{b}d\hat{b} - \frac{1}{2\pi}Ad\hat{b} - \frac{1}{4\pi}AdA$$

- For $A = 0$ we see that the Wilson-Fisher theory coupled to $U(1)_1$ is dual to itself with $U(1)_{-1}$; i.e. it is T-invariant.
  - This symmetry is not manifest as $|\Phi|^2 \leftrightarrow -\left|\widehat{\Phi}\right|^2$.
- The coupling to the background A needs an AdA counterterm; i.e. T-invariance is anomalous for nonzero A.
- A monopole operator $\mathcal{M}_b$ is not $U(1)_b$ invariant, but $\Phi^+\mathcal{M}_b$ is. It is charged under $U(1)_A$ and has spin $\frac{1}{2}$.

A new boson/boson duality

- Deform the theory by $|\Phi|^2 \leftrightarrow -|\widehat{\Phi}|^2$
- With one sign: gapped, low energy Lagrangian is $-\frac{1}{4\pi} AdA$
  - Φ is massive. The Φ particle is charged under $U(1)_b$ and therefore it is charged under $U(1)_A$. Its spin is transmuted to $\frac{1}{2}$.
  - $\langle\widehat{\Phi}\rangle$ Higgses $U(1)_{\hat{b}}$. Vortex is charged under $U(1)_A$ and its spin is $\frac{1}{2}$.
- With the opposite sign (T-reversal) the description is exchanged between the two sides of the duality. The particles have spin $-\frac{1}{2}$. Low energy Lagrangian vanishes.

Suggests a new boson/fermion duality

This looks a lot like the theory of a free massless fermion

$$i\bar{\Psi}\not{D}_A\Psi \quad \leftrightarrow \quad |D_b\Phi|^2 - |\Phi|^4 + \frac{1}{4\pi}bdb + \frac{1}{2\pi}Adb$$

- Global $\mathrm{U}(1)_{\mathrm{A}}$ symmetry
- T-invariance with an anomaly $\frac{1}{4\pi}AdA$
- $\Psi \leftrightarrow \Phi^+\mathcal{M}_b$ is charged under $U(1)_A$ and has spin $\frac{1}{2}$.
- With mass deformation $\bar{\Psi}\,\Psi \leftrightarrow |\Phi|^2$ depending on the sign
  - the low energy theory is trivial or $-\frac{1}{4\pi}AdA$
  - a particle with spin $\pm\frac{1}{2}$

Suggests a new boson/fermion duality

Can also find it by boldly substituting $N = k = N_f = 1$ in the conjectured duality with generic $N,\ k,\ N_f$.

Assume: a free fermion coupled to a background A is dual to a gauged Wilson-Fisher fixed point with Chern-Simons interaction for the dynamical field b

Can reverse the logic above and derive the known particle/vortex duality.

$$i\bar{\Psi} \not{D}_A \Psi \quad \leftrightarrow \quad |D_b\Phi|^2 - |\Phi|^4 + \frac{1}{4\pi} bdb + \frac{1}{2\pi} Adb$$

- Here both sides of the duality need a spin structure (spinors in the LHS and odd Chern-Simons level in the RHS)
  - But if A is a spin_c connection,

$$\int \frac{dA}{2\pi} = \frac{1}{2} \int w_2 \text{mod}\, \boldsymbol{Z} \,,$$

there is no need for a spin structure on either side.

Derive many other dualities

Starting with

$$i\bar{\Psi} \not{D}_A \Psi \quad \leftrightarrow \quad |D_b \Phi|^2 - |\Phi|^4 + \frac{1}{4\pi} bdb + \frac{1}{2\pi} Adb$$

we can derive other dualities by changing the two sides:

- Add a Chern-Simons counterterm for the classical field A
- Gauge it by turning A into a dynamical field a and adding a new classical field.
- Use other dualities.
- Repeat.

Another boson/fermion duality

For example, derive:

$$|D_B\Phi|^2 - |\Phi|^4 \quad \leftrightarrow \quad i\bar{\Psi} \not{D}_a \Psi + \frac{1}{2\pi} B da - \frac{1}{4\pi} B dB$$

LHS Wilson-Fisher fixed point

RHS QED with a single fermion, a.k.a $U(1)_{1/2}$

- Derived from the other duality
- Neither side needs a spin structure when a is a spin_c connection
- Need a Chern-Simons counterterm for B
- Can map the operators and check the phases
- RHS is T-reversal invariant. It acts non-locally. Quantum symmetry

A fermion/fermion duality

Derive: $i\bar{\Psi} \not{D}_A \Psi \quad \leftrightarrow$

$$i\bar{\chi}\not{D}_a\chi - \frac{2}{4\pi}bdb + \frac{1}{2\pi}adb + \frac{1}{2\pi}Adb - \frac{1}{4\pi}AdA$$

LHS free fermion

RHS QED with a single fermion, coupled to $U(1)_{-2}$ of b.

- If we incorrectly integrate out b, we find the previously mentioned version with improperly quantized Chern-Simons terms.
- No need for a spin structure when a and A are spin_c connections.
- Can map the operators and check the phases
- T-reversal invariance (with anomaly) is manifest in LHS. It acts non-locally in the RHS (quantum symmetry).

Repeat with $SO(N)$ and $SP(N)$

For example, the general $SO(N)$ duality leads for $k = N_f = 1$ to

$$i\psi \partial\!\!\!/ \psi \quad \leftrightarrow \quad (D_b\phi)^2 - \phi^4 + \frac{1}{2\cdot 4\pi}\mathrm{Tr}(bdb + \frac{2i}{3}b^3)$$

ψ a real Majorana fermion

ϕ a real scalar in the vector of $SO(N)$ and we take N>2

b an $SO(N)_1$ gauge field

$\psi \quad \leftrightarrow \quad \mathcal{M}_b$ is a $\boldsymbol{Z}_2$ monopole operator of $SO(N)_1$

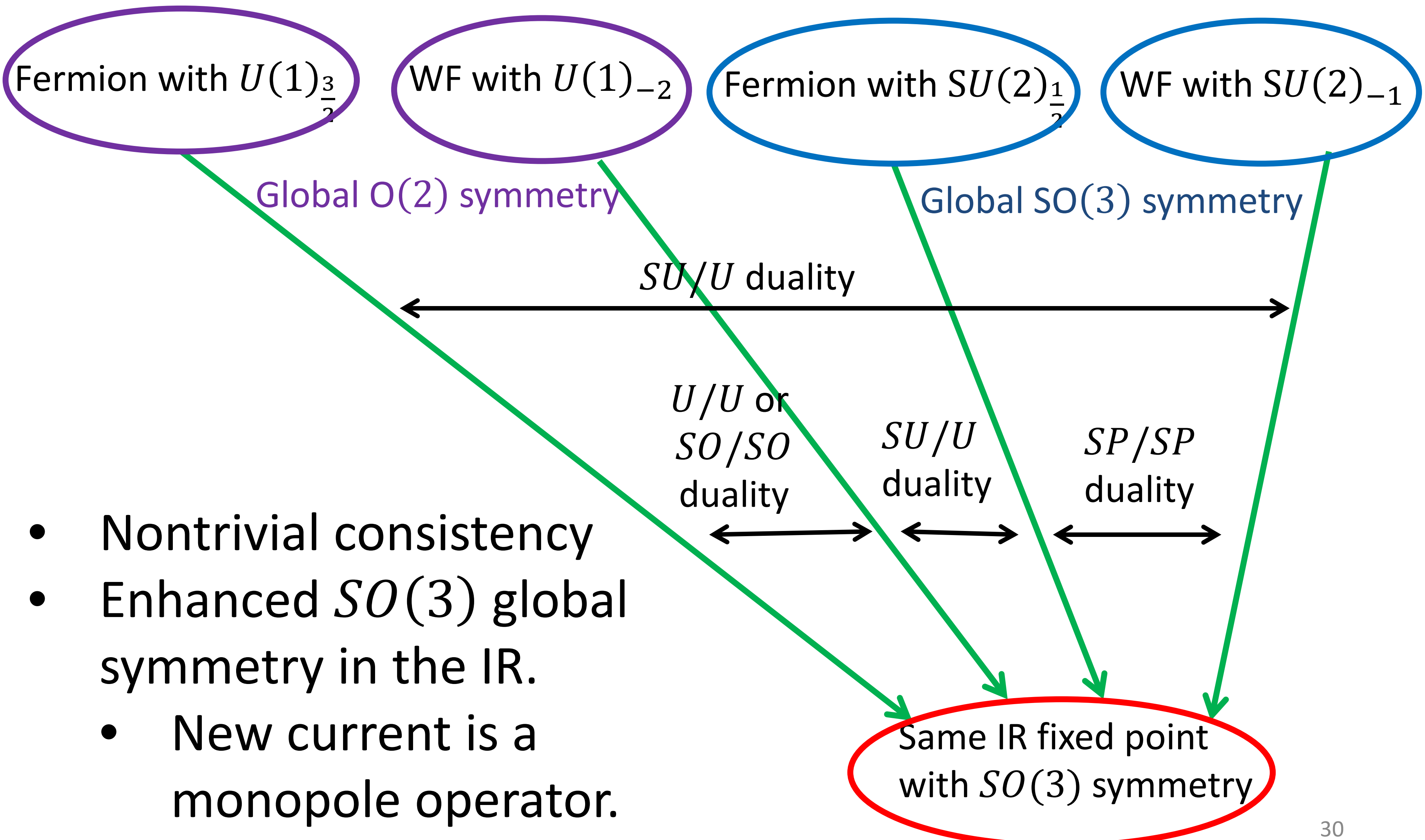
Charge conjugation $\leftrightarrow$ monopole number

Deformation by $\psi\psi \quad \leftrightarrow \quad \phi^2$ depends on the sign

Time reversal symmetry with gravitational anomaly. It is nonlocal in the RHS

Also in [Metlitski, Vishwanath, Xu]

Example of a web of dualities



- Nontrivial consistency
- Enhanced $SO(3)$ global symmetry in the IR.
  - New current is a monopole operator.

Review of special cases

$U(1)_0$ with $\Phi \leftrightarrow \widehat{\Phi}$ standard particle/vortex duality

$U(1)_1$ with $\Phi \leftrightarrow \Psi$ bosonization/fermionization

$U(1)_2$ with $\Phi \leftrightarrow \ldots$ quantum $SO(3)$ global symmetry

In all these examples the scalars have a $|\Phi|^4$ interaction and we tune to a nontrivial fixed point

The monopole operator of $U(1)_k$ has spin $\frac{k}{2}$. It is $\widehat{\Phi}$, a free fermion Ψ, and a conserved current in these cases.

More

- ✓ Add gravitational Chern-Simons counterterms (more checks)
- ✓ Generalization to arbitrary N and k
  - Using a precise version of level/rank duality
  - Problem with large N_f
- ✓ Many more dualities and relations between them
- ✓ ‘t Hooft anomaly matching conditions in $2+1d$
- ❑ Much more can be done

Summary

- Duality is ubiquitous in physics.
- Ideas in different branches of physics are useful in other branches. The unity of physics.
- A rich duality web in $2+1d$ physics. Assuming some of these dualities leads to others. Many cross-checks.
  - Boson/boson
  - Boson/fermion
  - Fermion/fermion
- Many more dualities are likely to be present
  - Use the same techniques to find others
  - Find new dualities that do not follow trivially from these