Confinement, De-confinement, and $3d$ Topological Quantum Field Theory

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IAS

$SU(N)$ pure gauge theory in $4d$
Lore

$$\frac{1}{4g^2}\int Tr\,(F\wedge * F) + \frac{i\,\theta}{8\pi^2}\int \mathrm{Tr}(F\wedge F)$$

- θ is 2π-periodic (on a closed manifold)
- Generic θ
  - Unique vacuum (no TQFT at low energies), gapped spectrum
  - Confinement: $\langle W\rangle = \langle \mathrm{Tr}\, e^{i\oint a}\rangle \sim e^{-Area} \to 0$ at long distances
- θ multiple of π: $\mathcal{CP}$ symmetry (equivalently, time-reversal)
- θ odd multiple of π
  - $\mathcal{CP}$ is spontaneously broken – 2 vacua
  - Domain walls

$SU(N)$ pure gauge theory in $4d$
Additional observables

[Kapustin, NS; Gaiotto, Kapustin, NS, Willett]

- Study $PSU(N)$ bundles that are not $SU(N)$ bundles – nontrivial w_2 of the bundle.
- Can describe using a $\mathbb{Z}_N$ classical background (two-form) gauge field B that sets $w_2 = B$.
- B can be interpreted as a classical background gauge field of a $\mathbb{Z}_N$ one-form global symmetry. More below.
- This is not a $PSU(N)$ gauge theory, where B is summed over. More below.

$SU(N)$ pure gauge theory in $4d$
The operators [...; Kapustin, NS]

- Wilson lines $W(C) = \mathrm{Tr}(e^{i\oint_C a})\, e^{i\int_\Sigma B}$ with $C = \partial\Sigma$
- Charges: topological surface operators $U_E(X) = e^{i\oint_X u(a)}$
- The 't Hooft operator T is not a genuine line operator.
  - Since a Wilson line can detect the Dirac string emanating from the 't Hooft operator, the Dirac string is visible and sweeps a surface Σ

    $$T(C)e^{i\int_\Sigma u(a)}$$

    It is an open version of U_E.
- U_E can be interpreted as the worldsheet of a Dirac string.

$SU(N)$ pure gauge theory in $4d$
B-dependent counterterm

[Kapustin, NS; Gaiotto, Kapustin, NS, Willett; Gaiotto, Kapustin, Komargodski, NS]

- Can add to the action a counterterm in the classical fields:

$$\frac{2\pi\, i\, p}{2N}\int \mathcal{P}(B)$$

  $p = 1,2, \ldots, 2N,\ \ pN \in 2\mathbb{Z}$ ($\mathcal{P}$ is the Pontryagin square)

- For nonzero B there are fractional instantons and therefore θ is not 2π-periodic. Instead, $(\theta, p) \sim (\theta + 2\pi, p + N - 1)$
- $\theta \to \theta + 2\pi$ leads to different theories
  - Different contact terms
  - Different behavior on boundaries

$SU(N)$ pure gauge theory in $4d$
$\mathcal{CP}$ at $\theta = \pi$ [Gaiotto, Kapustin, Komargodski, NS]

$$\frac{i\,\theta}{8\pi^2}\int \mathrm{Tr}(F\wedge F) \;+\; \frac{2\pi\, i\, p}{2N}\int \mathcal{P}(B)$$
$$(\theta, p) \sim (\theta + 2\pi, p + N - 1)$$

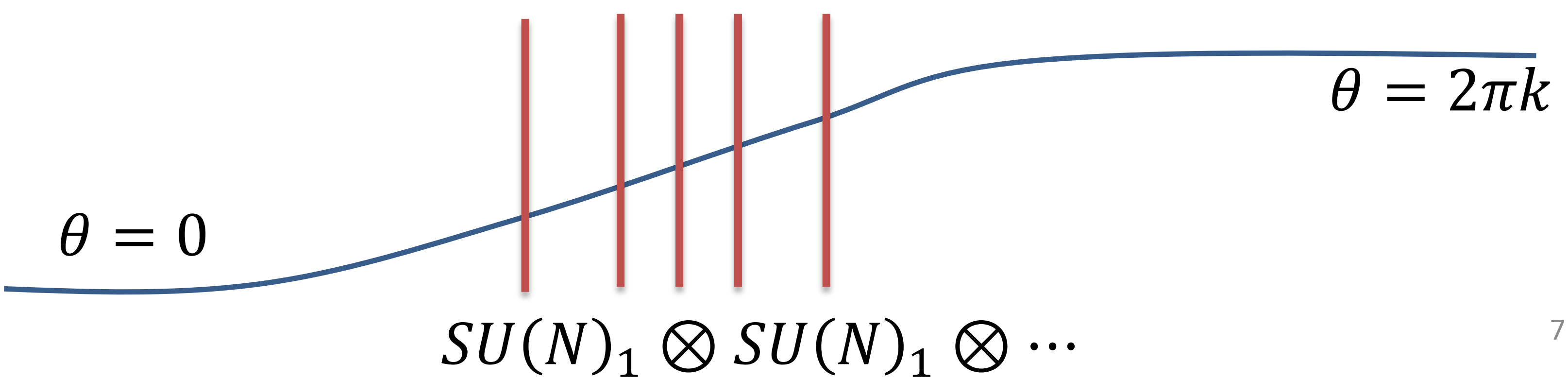
$$\mathcal{CP}\colon (\pi, p) \to (-\pi, -p) \sim (\pi, -p + N - 1)$$

- For N even, no value of p preserves the symmetry – mixed anomaly between $\mathcal{CP}$ and the $\mathbb{Z}_N$ one-form symmetry
- CP is spontaneously broken with 2 vacua
- Nontrivial domain wall between the two vacua at $\theta = \pi$
  - Anomaly inflow from the bulk
  - $SU(N)_1$ Chern-Simons theory on the wall.

$SU(N)$ pure gauge theory in $4d$
Interface [Gaiotto, Kapustin, Komargodski, NS]

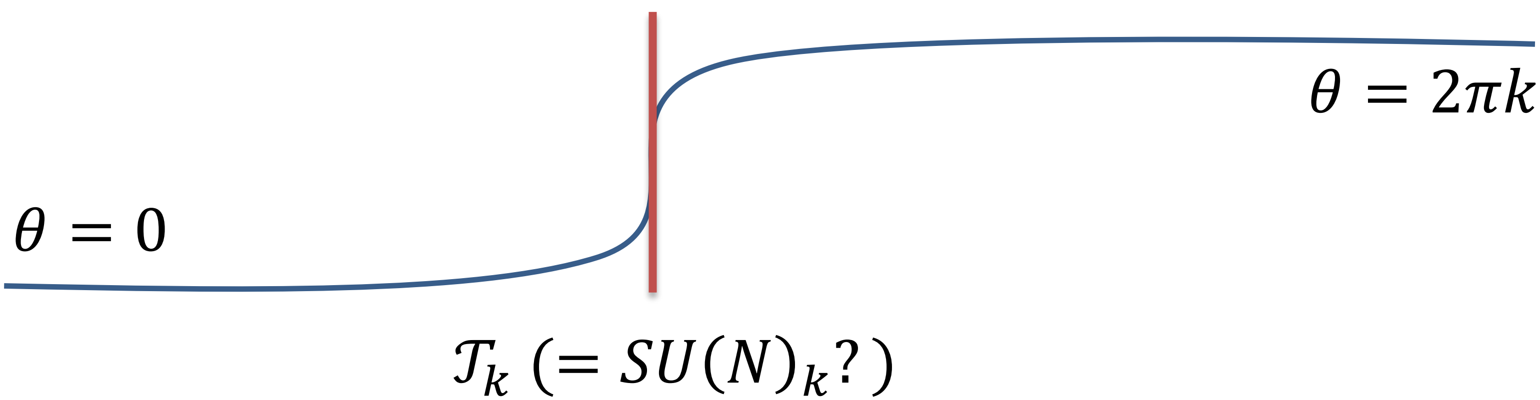
More generally, consider a space-dependent θ interpolating between $\theta = 0$ and $\theta = 2\pi k$ for some integer k

- If the interpolation is very slow (compared with the confinement scale of the theory), we have k copies of $SU(N)_1$ localized where θ crosses an odd multiple of π.



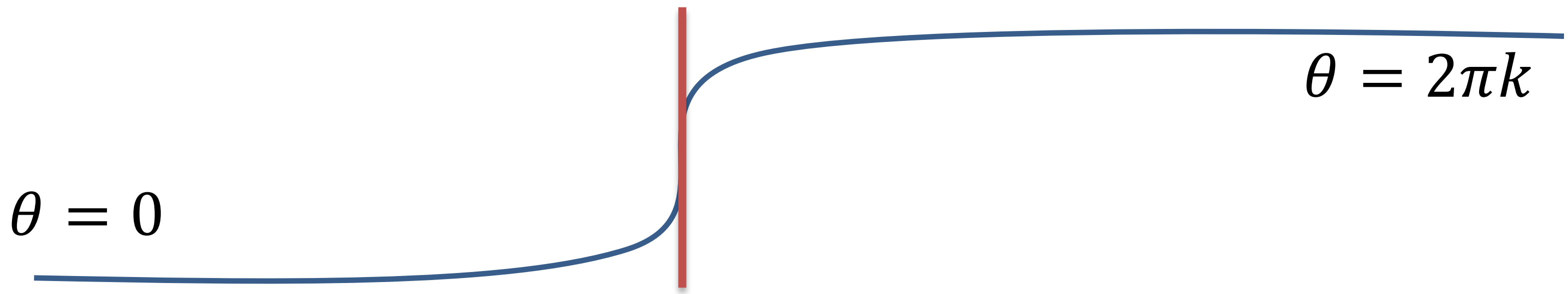
$SU(N)$ pure gauge theory in $4d$
Interface

- If the interpolation is fast compared with the confinement scale of the theory, we can have another TQFT $\mathcal{T}_k$.
- Need better dynamical control to determine $\mathcal{T}_k$. One option is $\mathcal{T}_k = SU(N)_k$. This is consistent with the anomaly flow from the bulk.



- If the interface is sharp (discontinuous), the result is not universal, but it must have the same anomaly.

$SU(N)$ pure gauge theory in $4d$
Interface



$\mathcal{T}_k \; (= SU(N)_k ?)$

- Can think of it also as separating regions with the same θ, but different values of p.

$\theta = 0$
p^-

$\theta = 0$
$p^+ = p^- - k(N-1)$

$\mathcal{T}_k \; (= SU(N)_k ?)$

$SU(N)$ pure gauge theory in $4d$

Interface

$\theta = 0$

p^-

$\theta = 0$

$p^+ = p^- - k(N-1)$

$\mathcal{T}_k\ (= SU(N)_k?\,)$

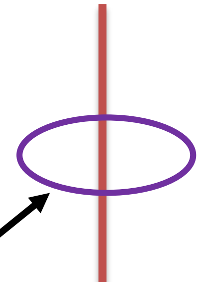
- On one side of the interface monopoles condense and lead to confinement. On the other side dyons condense and lead to "oblique confinement."
- By continuity, none of them condense on the interface and hence no confinement there.
  - The $SU(N)_k$ Wilson lines are Wilson lines of the microscopic gauge theory.
  - Surprise: they have nontrivial braiding – probe quarks are anyons.

$SU(N)$ pure gauge theory in $4d$

Interface [Hsin, Lam, NS]

$\theta = 0$

p^-



$\theta = 0$

$p^+ = p^- - k(N-1)$

$\mathcal{T}_k \; (= SU(N)_k ?)$

- Consider a $\mathbb{Z}_N$ charge operator $U_E(X) = e^{i \oint_X u(a)}$ that pierces the interface.
- We interpreted it as the worldsheet of a Dirac string.
- It is associated with a monopole on one side and a dyon on the other side. Therefore, it has electric charge k on the wall. It is a Wilson line there.
- This explains why there is braiding between Wilson lines on the wall.

$PSU(N)$ pure gauge theory in $4d$
Gauge the $\mathbb{Z}_N$ one-form global symmetry

[Kapustin, NS; Gaiotto, Kapustin, NS, Willett]

- Make B dynamical and denote it by b. This amounts to summing over w_2 of the $PSU(N)$ bundles
- The Wilson line is not a genuine line operator
$$W(C,\Sigma) = \mathrm{Tr}(e^{i\oint_C a})\, e^{i\int_\Sigma b}$$
with $C = \partial\Sigma$
- The correlation functions of $U_E(X) = e^{i\oint_X u(a)}$ are trivial.
- For $p = 0$ the 't Hooft operator is a genuine line operator
$$T(C)e^{i\int_\Sigma u(a)}$$
because there is no dependence on Σ. (For other values of p it should be multiplied by a Wilson line.)

$PSU(N)$ pure gauge theory in $4d$
Gauge the $\mathbb{Z}_N$ one-form global symmetry

- New surface operator

$$U_M = e^{i \oint_X b} = \mathrm{e}^{\mathrm{i} \oint_X w_2}$$

  - It generates a magnetic $\mathbb{Z}_N$ one-form symmetry
  - $\langle U_M(X)\, T(C) \rangle = e^{\frac{2\pi i}{N} \langle X, C \rangle} \langle T(C) \rangle$
    $\langle X, C \rangle$ the linking number.
  - The Wilson line $W(C,\Sigma) = \mathrm{Tr}(e^{i \oint_C a})\, e^{i \int_\Sigma b}$ is an open version of U_M

$PSU(N)$ pure gauge theory in $4d$
Dynamics

[Aharony, Tachikawa, NS; Kapustin, NS; Gaiotto, Kapustin, NS, Willett]

$$\frac{i\,\theta}{8\pi^2}\int \mathrm{Tr}(F\wedge F)\;+\;\frac{2\pi\,i\,p}{2N}\int \mathcal{P}(b)$$

$$(\theta,p)\sim(\theta+2\pi,p+N-1)$$

Now the lack of 2π-periodicity in θ is more important.

- For $p=0$ monopoles condense, the ‘t Hooft operator has a perimeter law
  - Gapped spectrum, but a nontrivial TQFT at low energies (not merely an SPT phase) – a $\mathbb{Z}_N$ gauge theory
- More generally, the low energy theory is a $\mathbb{Z}_L$ gauge theory (could be twisted on nonspin manifolds) with

$$L=\gcd(p,N)$$

$PSU(N)$ pure gauge theory in $4d$
Interface

In the $SU(N)$ theory

$$\theta = 0 \qquad p^-$$

$$\theta = 0 \qquad p^+ = p^- - k(N-1)$$

$\mathcal{T}_k \; (= SU(N)_k ?)$

In the $PSU(N)$ theory

$$\theta = 0 \qquad p^-$$

$\mathbb{Z}_{L^-}$ gauge theory

$L^\pm = \gcd(p^\pm, N)$

$$\theta = 0 \qquad p^+ = p^- - k(N-1)$$

$\mathbb{Z}_{L^+}$ gauge theory

???

Cannot have $\frac{\mathcal{T}_k}{\mathbb{Z}_N} \; (= PSU(N)_k ?)$ on the interface – it is not consistent!

$PSU(N)$ pure gauge theory in $4d$
Interface

$\theta = 0$
p^-
$\mathbb{Z}_{L^-}$ gauge theory

???

$\theta = 0$
$p^+ = p^- - k(N-1)$
$\mathbb{Z}_{L^+}$ gauge theory

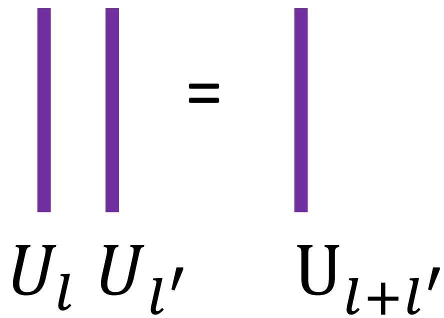
$$L^\pm = \gcd\left(p^\pm, N\right)$$

In order to figure out the theory on the interface we need to understand better the one-form global symmetry, its anomaly, and its gauging.

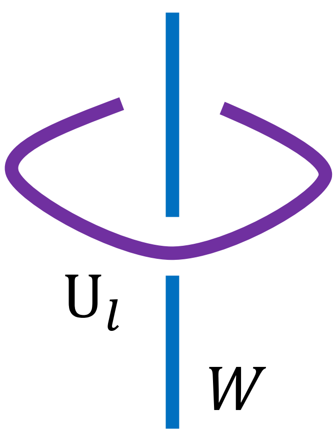
A $3d$ TQFT $\mathcal{T}$ with a $\mathbb{Z}_N$ one-form global symmetry [Gaiotto, Kapustin, NS, Willett]

There are line operators U_l with l integer modulo N such that

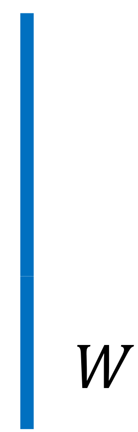
- $U_l U_{l'} = U_{l+l'}$



- Any line $W \in \mathcal{T}$ has a $\mathbb{Z}_N$ charge $q(\mathrm{W})$ ($\mathbb{Z}_N$ representation)



$$= e^{\frac{2\pi i q(W) l}{N}}$$



A $3d$ TQFT $\mathcal{T}$ with a $\mathbb{Z}_N$ one-form global symmetry [Gaiotto, Kapustin, NS, Willett]

Examples

- $SU(N)_k$ has a $\mathbb{Z}_N$ one-form symmetry associated with the center of the gauge group. It is generated by a line in a representation of k symmetric fundamentals.
- $U(1)_N$ has N lines (for N even) realizing a $\mathbb{Z}_N$ one-form symmetry, generated by the line of charge one.
- $\mathbb{Z}_N$ gauge theory has a $\mathbb{Z}_N \otimes \mathbb{Z}_N$ one-form symmetry (electric and magnetic)

A $3d$ TQFT $\mathcal{T}$ with a $\mathbb{Z}_N$ one-form global symmetry [Gomis, Komargodski, NS; Hsin, Lam, NS]

Consistency implies that the spins of the charge lines are

$$h(U_l) = \frac{p\, l^2}{2N} \bmod 1$$

Here $p = 0,1, \ldots, 2N$, $pN \in 2\mathbb{Z}$.

(In a spin TQFT ignore the second condition and $p \sim p + N$. By changing the $\mathbb{Z}_N$ generator we can relate different values of p.)

Using the braiding we interpret

$$p \bmod N = q(U_1)$$

as the $\mathbb{Z}_N$ charge of the generator of the $\mathbb{Z}_N$ symmetry.

If $p \neq 0$, we cannot gauge the one-form symmetry. p characterizes the ‘t Hooft anomaly of the symmetry.

A $3d$ TQFT $\mathcal{T}$ with a $\mathbb{Z}_N$ one-form global symmetry

$$h(U_l) = \frac{p\, l^2}{2N} \bmod 1$$

p characterizes the ‘t Hooft anomaly of the symmetry.

Coupling to a background $\mathbb{Z}_N$ gauge field B the corresponding anomaly is our $4d$ bulk term

$$\frac{2\pi\, i\, p}{2N} \int \mathcal{P}(B)$$

A $3d$ TQFT $\mathcal{T}$ with a $\mathbb{Z}_N$ one-form global symmetry with $p = 0$

$$h(U_l) = \frac{p\, l^2}{2N} \bmod 1$$

For $p = 0$

- the lines U_l are $\mathbb{Z}_N$ neutral
- their spins vanish modulo an integer
- their braiding is trivial
- there is no ‘t Hooft anomaly
- We can gauge the symmetry

Gauge the $\mathbb{Z}_N$ one-form global symmetry when $p = 0$ [Moore, NS]

For $p = 0$ we can gauge the symmetry (known in the condensed matter literature as "anyon condensation")

- Remove from $\mathcal{T}$ all the $\mathbb{Z}_N$ charged lines ($q(W) \neq 0 \bmod N$)
- Identify the lines $W \sim U_1 W$
- If a line W is the same as $U_1 W$, it appears multiple times

For our problem with the interface in $PSU(N)$ we need to gauge a TQFT with nonzero p.

A $3d$ TQFT $\mathcal{T}$ with a $\mathbb{Z}_N$ one-form global symmetry with $p \neq 0$

$$h(U_l) = \frac{p\, l^2}{2N} \bmod 1$$

For $p = N$ we can use essentially the standard gauging (except the identification) to find a spin TQFT.

For $L = \gcd(p, N) \neq N$ the lines $l = \frac{N}{L}\, \hat{l}$ lead to a $\mathbb{Z}_L \subset \mathbb{Z}_N$ subgroup, whose p is $\hat{p} = \frac{pN}{L} = 0 \bmod L$.

It can be gauged as above.

The resulting theory has $\mathbb{Z}_{N'}$ one-form symmetry with anomaly p' with $N' = N/L,\, p' = p/L$ and hence $L' = \gcd(N', p') = 1$.

So how should we deal with $L = \gcd(p, N) = 1$?

A $3d$ TQFT $\mathcal{T}$ with a $\mathbb{Z}_N$ one-form global symmetry with $L = \gcd(p, N) = 1$ [Hsin, Lam, NS]

$$h(U_l) = \frac{p\, l^2}{2N} \bmod 1$$

For every $W \in \mathcal{T}$ with charge $q(W)$ the line $U_1^r W = W'$ (need to show that W' is unique) with $rp + q(W) = 0 \bmod N$ is $\mathbb{Z}_N$ neutral.

- Hence, $\mathcal{T} = \mathcal{T}' \otimes \mathcal{A}^{N,p}$
  - $\mathcal{T}'$ includes all the neutral lines W'
  - $\mathcal{A}^{N,p}$ is a minimal TQFT with $\mathbb{Z}_N$ symmetry with anomaly p.
  - The factorization is also guaranteed by a theorem of [Muger; Drinfeld, Gelaki, Nikshych, Ostrik]
- This is quite surprising. All the information about the symmetry is in a decoupled universal sector $\mathcal{A}^{N,p}$!

The minimal $3d$ TQFT with a $\mathbb{Z}_N$ one-form global symmetry with anomaly p with $\gcd(p,N)=1$
$\mathcal{A}^{N,p}$ [Moore, NS; Hsin, Lam, NS]

Examples

- $U(1)_N = \mathcal{A}^{N,1}$
- $SU(N)_1 = \mathcal{A}^{N,N-1}$
- $U(1)_{Np} = \mathcal{A}^{N,p} \otimes \mathcal{A}^{p,N}$ for $\gcd(p,N)=1$ (this generalizes $\mathbb{Z}_{Np} = \mathbb{Z}_N \otimes \mathbb{Z}_p$)

$\mathcal{A}^{N,p} \otimes \mathcal{A}^{N,-p}$ has N^2 lines with an anomaly free diagonal $\mathbb{Z}_N$ one-form symmetry.

Gauging it leads to $(\mathcal{A}^{N,p} \otimes \mathcal{A}^{N,-p})/\mathbb{Z}_N$, which is a trivial theory.

Using $\mathcal{A}^{N,p}$ [Hsin, Lam, NS]

Starting with a theory $\mathcal{T}$ with a $\mathbb{Z}_N$ one-form symmetry with anomaly p such that $\gcd(p, N) = 1$, we cannot gauge the symmetry.

However, since

$$\mathcal{T} = \mathcal{T}' \otimes \mathcal{A}^{N,p}$$

we can extract $\mathcal{T}'$either by dropping $\mathcal{A}^{N,p}$, or by tensoring another factor and then gauging an anomaly free $\mathbb{Z}_N$ symmetry

$$\mathcal{T}' = (\mathcal{T} \otimes \mathcal{A}^{N,-p})/\mathbb{Z}_N$$

Since $\mathcal{A}^{N,p}$ is minimal, this procedure is canonical.

Back to the interface in $4d\ \ PSU(N)$

[Hsin, Lam, NS]

$\theta = 0$ p^- $\mathbb{Z}_{L^-}$ gauge theory	???	$\theta = 0$ $p^+ = p^- - k(N-1)$ $\mathbb{Z}_{L^+}$ gauge theory

$L^\pm = \gcd(p^\pm, N)$

For simplicity, let $L^\pm = 1$.

Then the bulk theories are trivial and there must be a *3d* TQFT on the interface.

(The analysis for generic $L^\pm$ is more subtle and is explained in the paper.)

Back to the interface in $4d\ \ PSU(N)$

[Hsin, Lam, NS]

$\theta = 0$
p^-

$\theta = 0$
$p^+ = p^- - k(N-1)$

$???$

For simplicity, let $L^\pm = \mathrm{gcd}\left(p^\pm, N\right) = 1.$

$$\mathcal{T}_k \;\; \rightarrow \;\; \frac{\mathcal{T}_k \otimes \mathcal{A}^{N,-p^-} \otimes \mathcal{A}^{N,p^+}}{\mathbb{Z}_N}$$

Since $p^+ - p^- = -k(N-1) = -\,p(\mathbb{Z}_N \subset \mathcal{T}_k)$, the diagonal $\mathbb{Z}_N$ in the numerator is anomaly free and can be gauged.

Can interpret $\mathcal{A}^{N,-p^-} \otimes \mathcal{A}^{N,p^+}$ as arising from the bulk on the left and the right such that we can perform the gauging.

Conclusions

4d $SU(N)$ gauge theory

- For generic θ the spectrum is gapped with a trivial low−energy theory and at $\theta = \pi$ there are two vacua.
- $\mathbb{Z}_N$ one form global symmetry
  - It is unbroken (the theory is confining)
  - We can couple the theory to a background two-form $\mathbb{Z}_N$ gauge field B and add a counterterm $\frac{2\pi\, i\, p}{2N} \int \mathcal{P}(B)$
  - Keeping track of this term, θ is $2\pi N$-periodic ($4\pi N$-periodic for even N on a non-spin manifold).
- Steep interface from $\theta = 0$ to $\theta = 2\pi k$ has a TQFT (e.g. $SU(N)_k$) on it

Conclusions

4d $PSU(N)$ gauge theory is obtained by gauging the $\mathbb{Z}_N$ one-form global symmetry of the $SU(N)$ theory.

- The low energy theory is a $\mathbb{Z}_L$ gauge theory with $L = \gcd(p, N)$
- Interfaces have more subtle TQFTs on them

Conclusions

A *3d* TQFT with a $\mathbb{Z}_N$ one-form global symmetry

- It is characterized by an integer $p \bmod 2N$, which determines
  - The charge of the generating line
  - The spins of the charge lines
  - The 't Hooft anomaly
- For $\gcd(p, N) = 1$ there is a minimal TQFT with $\mathbb{Z}_N$ one-form symmetry and anomaly p, $\mathcal{A}^{N,p}$
  - Any theory $\mathcal{T}$ with a $\mathbb{Z}_N$ one-form symmetry with anomaly p, such that $\gcd(p, N) = 1$, factorizes $\mathcal{T} = \mathcal{T}' \otimes \mathcal{A}^{N,p}$