Anomalies in the Space of Coupling Constants

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Introduction

- Long history of exploring a classical or quantum theory as a function of its parameters. Two complementary applications:
  - Family of theories (e.g. Berry phase)
  - Make the coupling constants spacetime dependent
- More generally, we should view every theory as a function of
  - Spacetime dependent coupling constants
  - Spacetime dependent background gauge fields for global symmetries
  - Spacetime geometry (and associated choices like spin-structure, etc.)
  - Explore defects (e.g. interfaces) as these vary in spacetime

Introduction

- Traditionally, we study anomalies in
  - global symmetries by coupling the system to spacetime dependent background gauge fields (these can be continuous or discrete and the symmetries can be ordinary symmetries or higher form symmetries)
  - the geometry by placing the system in arbitrary spacetime metric (and associated choices like spin-structure, etc.)
- Then, the anomaly can be described by a classical field theory in one higher dimension (and sometimes it is convenient to study it in two higher dimensions).
- Many applications starting with ‘t Hooft anomaly matching.
- Since we will study spacetime dependent coupling constants, the anomaly action can depend on them.

Introduction

This “generalized anomaly” has two classes of applications

- An anomaly in a (multi-parameter) family of theories can lead, à la ‘t Hooft, to constraints on its long-distance behavior, e.g. predict phase transitions.
  - This is also useful as a test of dualities
- The coupling constants can vary in spacetime leading to a defect. The anomaly constrains the dynamics on the defect.

Since this view unifies many different, recently-studied phenomena, some of the results may seem familiar to many of you.

Introduction

- This view will streamline, unify, and strengthen many known results, and will lead to new ones.
- We will start by demonstrating them in very simple examples (QM of a particle on a ring, 2d $U(1)$ gauge theory).
  - Since these examples are elementary, the more powerful formalism is not essential. But the examples provide a good pedagogical way to demonstrate the formalism.
- Then we will briefly turn to more advanced and more recently studied examples.
- We'll end by revisiting the free 4d fermion.

Quantum mechanics of a particle on a ring

$$\mathcal{L} = \frac{1}{2}\dot{q}^2 + \frac{i}{2\pi}\,\theta\dot{q} \qquad \text{with } q \sim q + 2\pi$$

$\theta \sim \theta + 2\pi$

Spectrum $E_n = \frac{1}{2}\left(n - \frac{\theta}{2\pi}\right)^2$

Symmetries:

- $U(1)$: $q \to q + \alpha$
- $\mathcal{CT}$: $q(\tau) \to -q(-\tau)$
- $\theta = 0, \pi$ also $\mathcal{C}$: $q \to -q$ (and therefore also $\mathcal{T}$) combining to $O(2)$
- $\theta = \pi$ the Hilbert space is in a projective representation.
  - A mixed $\mathcal{C}$-$U(1)$ anomaly [Gaiotto, Kapustin, Komargodski, NS]. This guarantees level-crossing there.

Quantum mechanics of a particle on a ring

[Gaiotto, Kapustin, Komargodski, NS]

Couple to a background $U(1)$ gauge field A

$$\mathcal{L} = \frac{1}{2}(\dot{q} + A)^2 + \frac{i}{2\pi}\,\theta(\dot{q} + A) + i\,kA$$

k is a coupling for the background field only – a counterterm.

With nonzero A the θ periodicity is modified

$$(\theta, k) \sim (\theta + 2\pi, k - 1)$$

- Related to that, the spectrum $E_n = \frac{1}{2}\left(n - \frac{\theta}{2\pi}\right)^2$ is invariant under $\theta \to \theta + 2\pi$, but the states are rearranged $|n\rangle \to |n + 1\rangle$.
- We can restore the 2π periodicity by adding a "bulk term" $\frac{i}{2\pi}\theta dA$.

Quantum mechanics of a particle on a ring

[Gaiotto, Kapustin, Komargodski, NS]

$$\mathcal{L} = \frac{1}{2}\dot{q}^2 + \frac{i}{2\pi}\,\theta\dot{q}$$

Break $U(1) \to \mathbb{Z}_N$ with a potential, e.g. $V(q) = \cos(Nq)$.

The qualitative conclusions are unchanged.

- For N even, we still have a similar $\mathcal{C} - \mathbb{Z}_N$ anomaly at $\theta = \pi$ and hence the ground state is degenerate there.
- For N odd, there is no anomaly – not a projective representation. But there is still degeneracy at $\theta = \pi$.
  - In terms of background fields, we can preserve $\mathcal{C}$ at $\theta = \pi$ by adding a counterterm $i\,\frac{N-1}{2}A$ for the $\mathbb{Z}_N$ gauge field A. But then there is no $\mathcal{C}$ at $\theta = 0$ (referred to as "global inconsistency").

Quantum mechanics of a particle on a ring

$$\mathcal{L} = \frac{1}{2}\dot{q}^2 + V(q) + \frac{i}{2\pi}\,\theta\dot{q}$$

Break $\mathcal{C}$ and $\mathcal{T}$ for all θ, e.g. by making $V(q)$ generic (but $\mathbb{Z}_N$ invariant) and adding degrees of freedom.

Now, the symmetry is only $\mathbb{Z}_N$ (no $U(1)$ and no $\mathcal{C}$).

Add a $\mathbb{Z}_N$ background field A and a counterterm ikA (with k an integer modulo N). Again

$$(\theta, k) \sim (\theta + 2\pi, k - 1)$$

Continuously shifting θ by 2π, the states are rearranged – a state with $\mathbb{Z}_N$ charge l is mapped to a state with a $\mathbb{Z}_N$ charge $(l+1) \mathrm{mod}\, N$.

The ground state must jump (level-crossing) at least once in $\theta \in [0,2\pi)$. There is a “phase transition.”

Quantum mechanics of a particle on a ring

$$\mathcal{L} = \frac{1}{2}\dot{q}^2 + V(q) + \frac{i}{2\pi}\,\theta\dot{q}$$
$$(\theta, k) \sim (\theta + 2\pi, k - 1)$$

Two views on the parameter space

- Either $\theta \sim \theta + 2\pi N$ and preserve the $\mathbb{Z}_N$ symmetry
- Or $\theta \sim \theta + 2\pi$, but violate the $\mathbb{Z}_N$ symmetry

Generalized anomaly between the $\mathbb{Z}_N$ symmetry and $\theta \sim \theta + 2\pi$ (related discussion in [Thorngren; NS, Tachikawa, Yonekura]).

The anomaly is characterized by the 2d bulk term $\frac{i}{2\pi}\int d\theta A$.
(Below it will be defined carefully.)

Quantum mechanics of a particle on a ring

Anomaly between the global $\mathbb{Z}_N$ symmetry and the θ periodicity.

It is characterized by the 2d bulk term $\frac{i}{2\pi}\int d\theta A$

- Viewing our system as a one-parameter family of theories labeled by θ the anomaly means that there must be level-crossing – "a phase transition." (Alternatively, this follows from tracking the $\mathbb{Z}_N$ charge of the ground state.)
- "Interfaces" where θ changes as a function of time carry $\mathbb{Z}_N$ charge.
- In order to discuss the interfaces we need to define the action more carefully.

Quantum mechanics of a particle on a ring

Differential Cohomology for pedestrians

When $\theta \in \mathbb{S}^1$ is Euclidean time dependent $\oint \theta(\tau)\dot{q}(\tau)d\tau$ should be defined more carefully.

Divide the Euclidean-time circle into patches $\mathcal{U}_I = (\tau_I, \tau_{I+1})$, where θ is a continuous function to $\mathbb{R}$ (no 2π jump) and it jumps by $2\pi m_I$ with $m_I \in \mathbb{Z}$ on overlaps of patches.

$$\frac{1}{2\pi}\oint \theta(\tau)\dot{q}(\tau)d\tau \equiv \left(\sum_I \frac{1}{2\pi}\int_{\mathcal{U}_I} \theta dq + m_I q(\tau_I)\right) \mod 2\pi$$

Quantum mechanics of a particle on a ring

Differential Cohomology for pedestrians

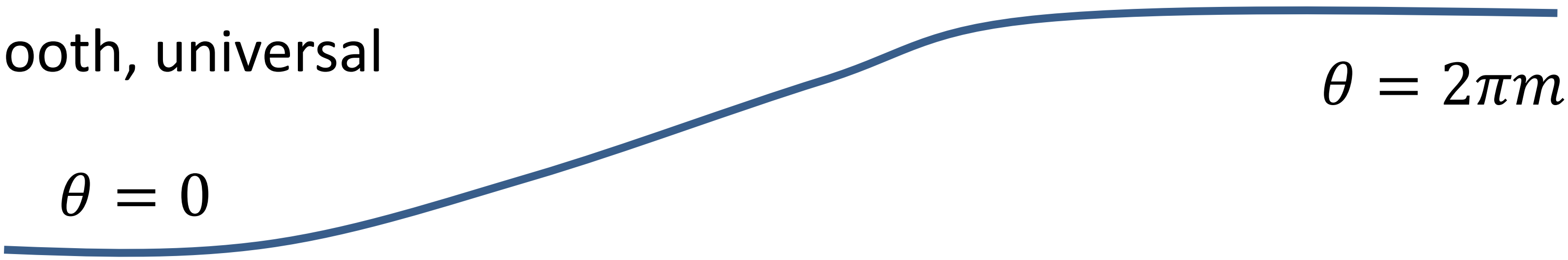
$$\frac{1}{2\pi}\oint \theta(\tau)\dot{q}(\tau)d\tau \equiv \left(\sum_I \frac{1}{2\pi}\int_{\mathcal{U}_I} \theta dq + m_I q(\tau_I)\right) \bmod 2\pi$$

- Independent of the trivialization
- Invariant under $q \to q + 2\pi$ and under $\theta \to \theta + 2\pi$
- If $\theta(\tau)$ has nonzero winding $M = \sum_I m_I$, this term is not invariant under $U(1)$: $q \to q + \alpha$.

This is our anomaly.

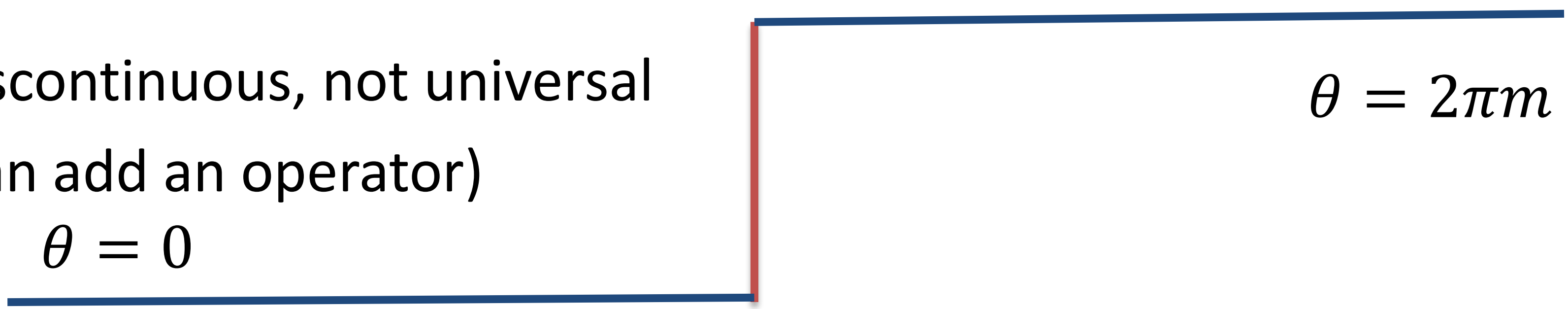
3 kinds of "interfaces"

Smooth, universal



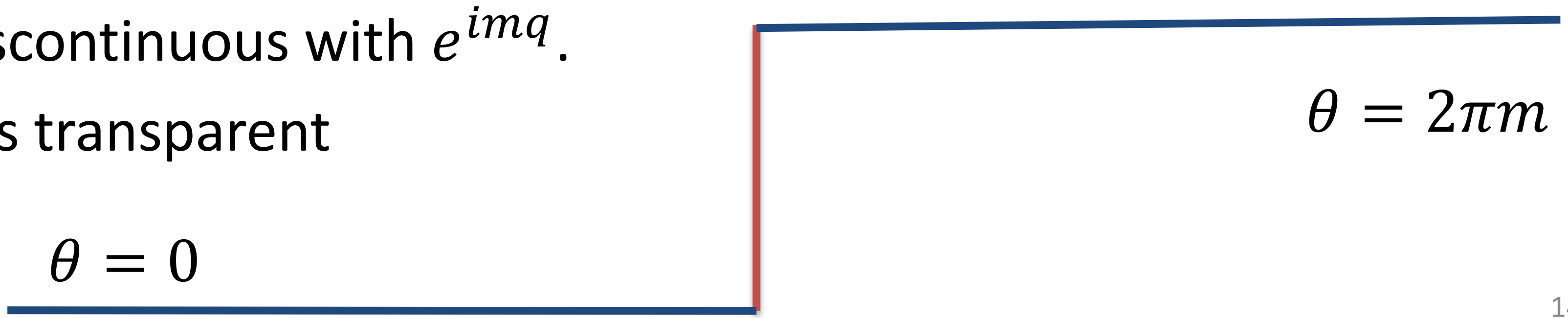
Discontinuous, not universal

(can add an operator)

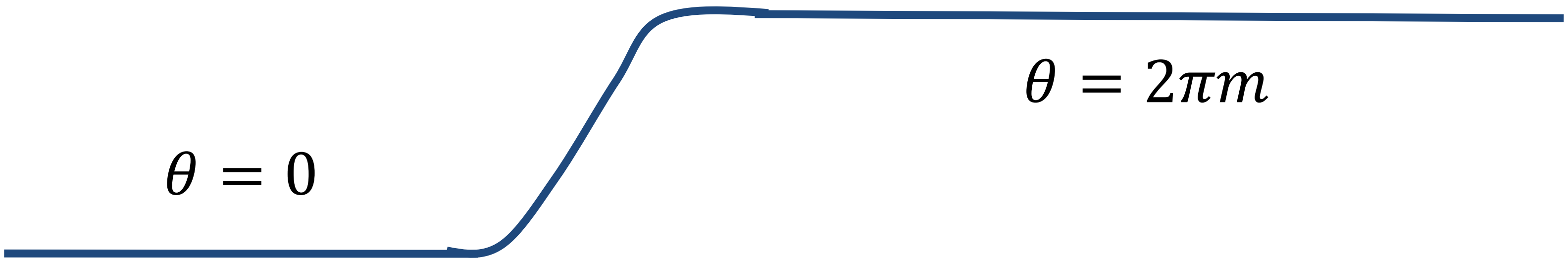


Discontinuous with e^{imq}.

It is transparent



A steep but smooth “interface”



This is a rapid change in the Hamiltonian at some Euclidean time. In the “sudden approximation” it relabels the states

$$|n\rangle \rightarrow |n+m\rangle,$$

which can be achieved by multiplying by e^{imq}. This means that the interface has $U(1)$ charge m.

When θ winds m times around the Euclidean circle, the $U(1)$ symmetry is violated by m unites.

This anomaly is related to a "symmetry"

Normally an anomaly is associated with a global symmetry: 0-form, 1-form, etc.

We can think of this anomaly as associated with a "-1-form symmetry."

Just as -1-branes (instantons) are not branes, a -1-form global symmetry is not a symmetry.

If we view it as a global symmetry,

- θ is its gauge field (its transition functions involve jumps by $2\pi\mathbb{Z}$)
- $d\theta$ is the field strength (well defined even across overlaps)
- Then, this anomaly follows the standard anomaly picture.

2d $U(1)$ gauge theory

Consider a 2d $U(1)$ gauge theory with N charge-one scalars ϕ^i and impose that the potential $V(|\phi|^2)$ is $SU(N)$ invariant.

Include $\frac{i}{2\pi}\theta da$.

For spacetime-dependent θ this leads to a background current $\frac{1}{2\pi}d\theta$.

Specifically, when θ depedends on space, there is background charge density.

And when θ winds around space

$$m = \frac{1}{2\pi}\oint d\theta$$

there is total background charge m.

2d $U(1)$ gauge theory

We are not going to assume charge conjugation symmetry.

- The system has a $PSU(N)$ global symmetry and we couple it to a background $PSU(N)$ gauge field B with the counterterm

$$2\pi i\ \frac{k}{N} w_2(B)$$

$w_2(B)$ is the second Stiefel-Whitney class of the $PSU(N)$ bundle. k is an integer modulo N.

- Now $(\theta, k) \sim (\theta + 2\pi, k - 1)$ [Gaiotto, Kapustin, Komargodski, NS].

2d $U(1)$ gauge theory

- Now $(\theta, k) \sim (\theta + 2\pi, k - 1)$ [Gaiotto, Kapustin, Komargodski, NS].
- This can be viewed as a mixed anomaly between the periodicity of θ and the global $PSU(N)$ symmetry.
- It is described by the 3d anomaly term (see also [Thorngren])

$$\frac{i}{N}\int d\theta w_2(B)$$

2d $U(1)$ gauge theory

$$\frac{i}{N}\int d\theta w_2(B)$$

Use the anomaly as in the QM example.

- Using the anomaly in a one-parameter family of theories: since the system is gapped, it must have a phase transition at some θ.
  - Note that we do not assume charge conjugation symmetry.
- A smooth interface, where $\theta \to \theta + 2\pi m$
  - carries background electric charge m
  - has an anomalous QM system on it – a projective $PSU(N)$ representation – an $SU(N)$ representation with N-ality m.
  - Can interpret as m charged particles due to Coleman's mechanism.

2d $U(1)$ gauge theory

$$\frac{i}{2\pi}\int \theta da$$

When θ is spacetime dependent and $\theta \sim \theta + 2\pi$ this integral needs to be defined more carefully.

As above, we divide spacetime to patches, integrate $\frac{i}{2\pi}\int \theta da$ in each patch. But add a "correction term" of the form $in\int a$ when θ is shifted by $2\pi n$ between two patches.

This guarantees that the answer is independent of the trivialization and hence makes it well defined when θ winds.

(We needed such a definition for the bulk term in the particle on the ring example.)

4d $SU(N)$ gauge theory

This system has a one-form $\mathbb{Z}_N$ global symmetry.

Couple it to a classical two-form $\mathbb{Z}_N$ gauge field B. It twists the $SU(N)$ bundle to a $PSU(N)$ bundle with [Kapustin, NS]

$$w_2(a) = B \; ,$$

where $w_2(a)$ is the second Stiefel-Whitney class of the $PSU(N)$ bundle – ‘t Hooft twisted configurations.

Mixed anomaly between the $\mathbb{Z}_N$ one-form symmetry and $\theta \sim \theta + 2\pi$ characterized by the 5d term

$$i\,\frac{N-1}{2N}\int d\theta\, \mathcal{P}(B)$$

with $\mathcal{P}(B)$ the Pontryagin square of B.

4d $SU(N)$ gauge theory

$$i\frac{N-1}{2N}\int d\theta\, \mathcal{P}(B)$$

This leads to earlier derived results:

- In $\mathcal{N}=1$ SUSY, a mixed anomaly between the $\mathbb{Z}_{2N}$ R-symmetry and the $\mathbb{Z}_N$ one-form symmetry. Hence, a TQFT on domain walls [Gaiotto, Kapustin, NS, Willett] in agreement with the string construction of [Acharya, Vafa].
- …

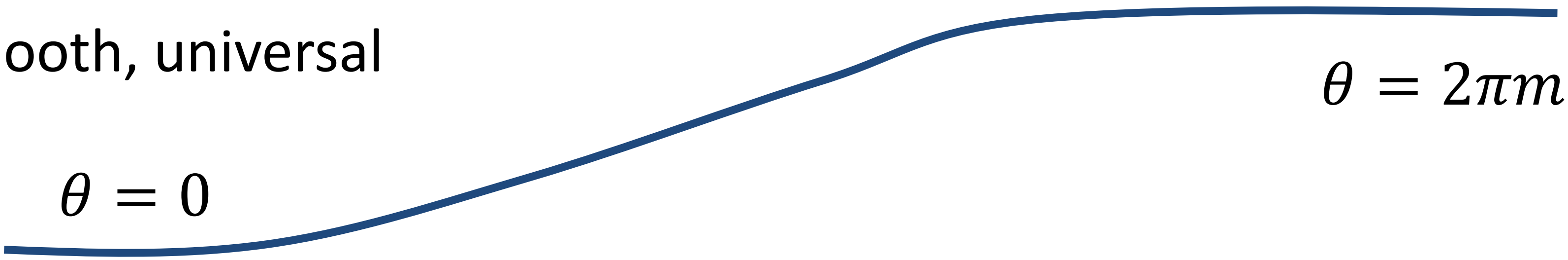
4d $SU(N)$ gauge theory

$$i\frac{N-1}{2N}\int d\theta\, \mathcal{P}(B)$$

- Without SUSY, a mixed anomaly between $\mathcal{T}$ (equivalently, $\mathcal{CP}$) and the $\mathbb{Z}_N$ one-form symmetry at $\theta = \pi$ (for even N) implies a transition at $\theta = \pi$ (or elsewhere) [Gaiotto, Kapustin, Komargodski, NS]
  - Can use the new anomaly in similar systems without $\mathcal{T}$, (e.g. add a massive scalar with coupling $i\phi\, Tr(F \wedge F)$). The gapped system must have a transition for some θ.
- Nontrivial TQFT on interfaces where θ changes by $2\pi m$ [Gaiotto, Komargodski, NS; Hsin, Lam, NS]

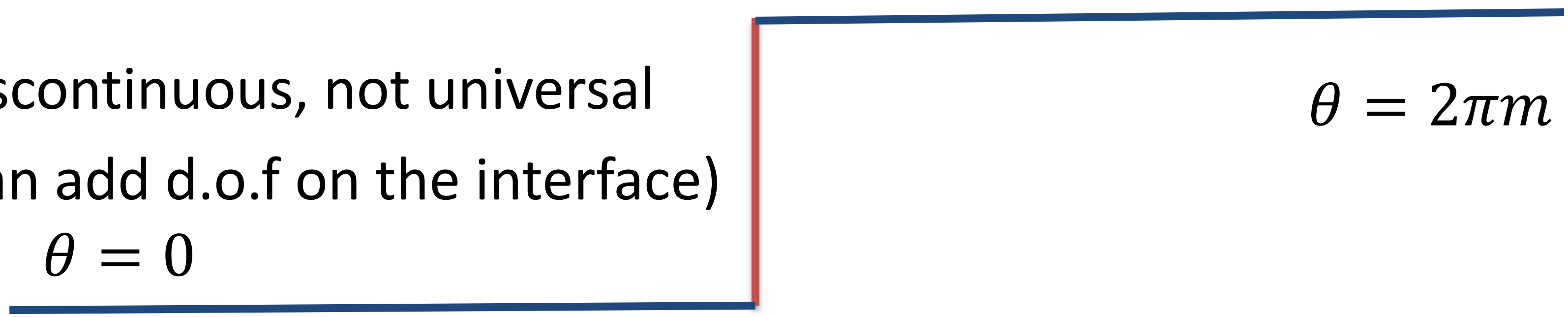
3 kinds of interfaces

Smooth, universal



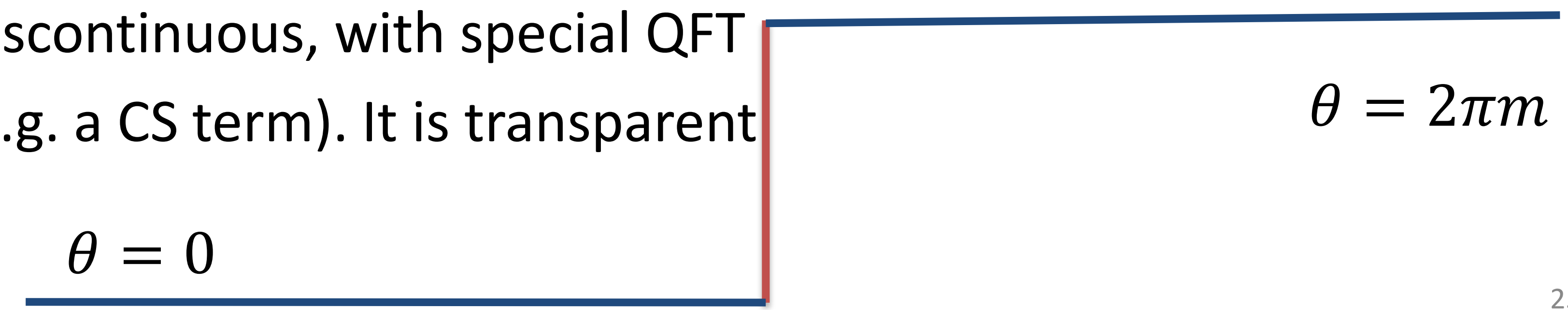
Discontinuous, not universal

(can add d.o.f on the interface)



Discontinuous, with special QFT

(e.g. a CS term). It is transparent



A free massive Weyl fermion in 4d

- The parameters: a complex mass m.
- Upon compactification of the m plane, a nontrivial 2-cycle in the parameter space.
- The phase of m can be removed by a chiral rotation $\psi \to e^{i\alpha}\psi$. But because of a $U(1)$-gravitational anomaly, the Lagrangian is shifted by $\frac{i\alpha}{192\pi^2} Tr(R \wedge R)$.
- As above, we should add the counterterm $\frac{i\,\theta_G}{384\,\pi^2} Tr(R \wedge R)$, and then we should identify $(m, \theta_G) \sim \left(e^{i\alpha}m, \theta_G + \alpha\right)$

A free massive Weyl fermion in 4d

$$(m, \theta_G) \sim \left(e^{i\alpha} m, \theta_G + \alpha\right)$$

- For nonzero m we can remove the anomaly, i.e. absorb the phase of m, in a redefinition: $\theta_G \to \theta_G - \arg(m)$. But we cannot do it for all complex m.
- This can be described as an anomaly using the 5d term

$$i \int \delta^{(2)}(m) d^2 m \, CS_g$$

with CS_g is the gravitational Chern-Simons term.

Equivalently, this can be written as the 6d term

$$\frac{i}{192\pi} \int \delta^{(2)}(m) d^2 m \, Tr(R \wedge R)$$

- Related discussion in [NS, Tachikawa, Yonekura].

A free massive Weyl fermion in 4d

$$\frac{1}{192\pi}\delta^{(2)}(m)d^2m\,Tr(R\wedge R)$$

Applications:

- A 2-parameter family of theories. At some point in the complex m plane the theory is not trivially gapped.
  - For the free fermion this is trivial. But the same conclusion in more complicated models, with additional fields and interactions. Note, we do not need any global symmetry.
- Defects. $m \sim (x+iy)$ is a string along the z axis. As in [Jackiw, Rossi], the anomaly means that there are Majorana-Weyl fermions ($c_L - c_R = \frac{1}{2}$) on the defect.
  - The same conclusion in a more complicated theory with more fields and interactions.

Conclusions

- To study ordinary anomalies we couple every global symmetry to a classical background field and place the theory in an arbitrary spacetime. Denote all these background fields by A. An anomaly is the statement that the partition function might not be gauge invariant or coordinate invariant.
- This is described by a higher dimensional classical (invertible) field theory for A.
- Similarly, we make all the coupling constants λ background fields and then the partition function might not be invariant under identifications in the space of coupling constants.
- This is described by a generalized anomaly theory – a higher dimensional classical (invertible) field theory for A and λ.

Conclusions

- This anomaly theory is invariant under renormalization group flow. (More precisely, it can be deformed continuously). Therefore, it can be used, à la ‘t Hooft, to constrain the long-distance dynamics and to test dualities. For example, it can force the low-energy theory to have some phase transitions.
- Defects are constructed by making A and λ spacetime dependent. Using these values in the anomaly theory, we find the anomaly of the theory along the defect.

Conclusions

- We presented examples of these anomalies (and their consequences) in various number of dimensions. The parameter space in the examples has one-cycles or two-cycles.
- We have studied many other cases with related phenomena.
  - 4d $SU(N), Spin(N), Sp(N)$ with quarks
- There are many other cases, we have not yet studied.