

A Duality Web in $2 + 1$
Dimensions
and the Unity of Physics

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IAS

Based on:

NS and E. Witten, [arXiv:1602.04251](https://arxiv.org/abs/1602.04251);

NS, T. Senthil, C. Wang, and E. Witten, [arXiv:1606.01989](https://arxiv.org/abs/1606.01989);

P.-S. Hsin, NS, [arXiv:1607.07457](https://arxiv.org/abs/1607.07457)

Recent related papers:

A. Karch and D. Tong, [arXiv:1606.01893](https://arxiv.org/abs/1606.01893);

J. Murugan and H. Nastase, [arXiv:1606.01912](https://arxiv.org/abs/1606.01912)

Three (almost) independent lines of development – the unity of physics

- The condensed matter, $3d$ quantum field theory route
- The supersymmetric route
- The AdS/CFT, large N route

The condensed matter, $3d$ QFT route

- Statistical transmutation: a massive particle coupled to a dynamical (statistical, emergent) gauge field with a Chern-Simons term can change its spin and statistics [[Wilczek](#); [Polyakov](#); [Jain](#)].
 - Many applications (FQHE, composite fermions, flux attachment, ...)
- This does not mean that a second-quantized theory of massless interacting bosons coupled to a gauge field with a Chern-Simons term is dual to a theory of fermions (or the other way around).

The condensed matter, 3d QFT route

Particle/vortex duality [Peskin; Dasgupta and Halperin]

$$|D_B \Phi|^2 - |\Phi|^4 \quad \leftrightarrow \quad |D_b \hat{\Phi}|^2 - |\hat{\Phi}|^4 + \frac{1}{2\pi} B db$$

- LHS is XY, $O(2)$ Wilson-Fisher.
- B is a background field coupled to a global $U(1)_B$ symmetry.
- RHS is a gauged version of this theory. b is a dynamical field.
- IR duality – two different theories flowing to the same IR fixed point.
- $\Phi \leftrightarrow \mathcal{M}_b$ is a monopole operator of b (charged under $U(1)_B$).
- $|\Phi|^2 \leftrightarrow -|\hat{\Phi}|^2$. Upon deformation: unbroken $U(1)_B$ phase is Higgs phase in the RHS; broken $U(1)_B$ phase massless b .

The condensed matter, 3d QFT route

Boson/fermion duality [Chen, Fisher, Wu; Barkeshli, McGreevy]

$$|D_B \Phi|^2 - |\Phi|^4 \quad \leftrightarrow \quad i\bar{\Psi} \not{D}_a \Psi + \frac{1}{2\pi} B da$$

LHS Wilson-Fisher fixed point (B is a background gauge field)

RHS QED with gauge field a with a single fermion, a.k.a $U(1)_{1/2}$

- Arguments involve elementary fields with fractional charges and fractional level Chern-Simons terms.
- LHS is T -reversal invariant, while RHS seems like it is not.
- LHS does not need a spin structure, while RHS does. Violating gravitational 't Hooft matching conditions?
- The IR behavior of the RHS is debated.

The condensed matter, 3d QFT route

Fermion/fermion duality [Son; Wang, Senthil; Metlitski, Vishwanath]

$$i\bar{\Psi} \not{D}_A \Psi \quad \leftrightarrow \quad i\bar{\chi} \not{D}_a \chi + \frac{1}{4\pi} A da$$

- Motivated by
 - physics of the lowest Landau level at half-filling [Halperin, Lee, Read]
 - T-Pfaffian state of topological insulators [Chen, Fidkowski, Vishwanath].
- Improperly quantized Chern-Simons term
- LHS is T -reversal invariant (with anomaly) and RHS seems like it is not. Its IR behavior is debated.

The supersymmetric route

- Many dualities of $4d \mathcal{N} = 1$ theories (IR dualities) [NS; ...]
- They motivated many dualities in $3d$
 - $\mathcal{N} = 2$ [Aharony, Hanany, Intriligator, NS, Strassler; Aharony; Gaiotto, Kutasov; ... Benini, Closset, Cremonesi; Intriligator, NS; Aharony, Razamat, NS, Willett; Park, Park; ...]
 - $\mathcal{N} = 4$ $3d$ mirror symmetry [Intriligator, NS; ...]
- These use particle/vortex duality
- Later derived by compactification of $4d \mathcal{N} = 1$ dualities on a circle and then flow with relevant operators [Aharony, Razamat, NS, Willett].
 - More checks
 - Leads to many new dualities

The supersymmetric route

- Many checks using supersymmetry and localization
- Related to string duality
- Connected to level/rank duality of $3d$ topological quantum field theory and $2d$ RCFT (rigorous [...; Hsin, NS]).
- Can flow from them to non-supersymmetric theories [Jain, Minwalla, Yokoyama; Gur-Ari, Yacoby; ...].
 - This motivates non-supersymmetric dualities.
 - But the flow might not be smooth.

The AdS/CFT, large N route

- Same $4d$ Vasiliev theory is dual to two different $3d$ field theories [Vasiliev ; Sezgin, Sundell; Klebanov, Polyakov; Giombi, Yin; Aharony, Gur-Ari, Yacoby; Giombi, Minwalla, Prakash, Trivedi, Wadia, Yin].
 - Scalars coupled to a Chern-Simons gauge theory
 - Fermions coupled to a Chern-Simons gauge theory
- Hence, a purely field theoretic duality between them
- Many explicit checks of this duality at large N [Maldacena, Zhiboedov; Aharony, Giombi, Gur-Ari, Maldacena, Yacoby; Jain, Minwalla, Sharma, Takimi, Wadia, Yokoyama; Minwalla, Yokoyama; Yokoyama; Jain, Mandlik, Minwalla, Takimi, Wadia, Yokoyama; Inbasekar, Jain, Mazumdar, Minwalla, Umesh, Yokoyama; ...]

Synthesis [Aharony]

3 [Aharony] + 1 [Hsin, NS] conjectures

N_f scalars at $|\Phi|^4$ point coupled to

N_f fermions coupled to

- $SU(N)_k \leftrightarrow U(k)_{-N+\frac{N_f}{2}, -N+\frac{N_f}{2}}$
- $U(N)_{k,k} \leftrightarrow SU(k)_{-N+\frac{N_f}{2}, -N+\frac{N_f}{2}}$
- $U(N)_{k,k+N} \leftrightarrow U(k)_{-N+\frac{N_f}{2}, -N-k+\frac{N_f}{2}}$
- $U(N)_{k,k-N} \leftrightarrow U(k)_{-N+\frac{N_f}{2}, -N+k+\frac{N_f}{2}}$

Fits the large N picture ($N, k \rightarrow \infty$ with finite N/k)

Fits the supersymmetric picture

Related to level/rank duality

Baryon and monopole operators match [Radicevic]

Puzzles

- Is it true for all N, k, N_f ?
- How can a theory of bosons, which does not need a spin structure be dual to a theory of fermions, which needs it?
- What is the relation to the dualities in the condensed matter literature (with puzzles about quantization of coefficients, T -reversal invariance, etc.)?
- What is the precise statement of the dualities (including the coupling to background gauge fields and their Chern-Simons counterterms)?
- Are the assumptions independent? Can we assume some of these dualities and derive others?
- Are there other such dualities?

Examine $N = k = N_f = 1$

Assume: a free fermion coupled to a background A is dual to a gauged Wilson-Fisher fixed point with Chern-Simons interaction for the dynamical field b

$$i\bar{\Psi} \not{D}_A \Psi \quad \leftrightarrow \quad |D_b \Phi|^2 - |\Phi|^4 + \frac{1}{4\pi} bdb + \frac{1}{2\pi} Adb$$

- Same symmetries
 - $U(1)_A$
 - T -reversal invariance (with anomaly) of the RHS follows from particle/vortex duality. T -reversal is a quantum symmetry there. (More below.)
- Conversely, assuming this duality we derive the known particle/vortex duality.

$$i\bar{\Psi} \not{D}_A \Psi \quad \leftrightarrow \quad |D_b \Phi|^2 - |\Phi|^4 + \frac{1}{4\pi} bdb + \frac{1}{2\pi} Adb$$

Mapping of operators

- $\Psi \leftrightarrow \Phi^+ \mathcal{M}_b$ with \mathcal{M}_b a monopole operator (charged under $U(1)_A$)
- Mass term $\bar{\Psi} \Psi \leftrightarrow |\Phi|^2$
- Mass deformation leads to two phases depending on the sign...

$$i\bar{\Psi}\not{D}_A\Psi \quad \Leftrightarrow \quad |D_b\Phi|^2 - |\Phi|^4 + \frac{1}{4\pi}bdb + \frac{1}{2\pi}Adb$$

- Mass deformation depends on the sign:
 - Φ is massive with spin $\frac{1}{2}$. It is charged under $U(1)_b$ and $U(1)_A$. It is mapped to the massive Ψ . (Statistical transmutation of massive particles.)
 - $U(1)_b$ is Higgsed. Vortex of spin $-\frac{1}{2}$ is charged under $U(1)_A$. It is mapped to the massive Ψ .
- T changes the sign of the fermion mass (= boson mass square) and maps Φ particles to vortices.

$$i\bar{\Psi} \not{D}_A \Psi \quad \Leftrightarrow \quad |D_b \Phi|^2 - |\Phi|^4 + \frac{1}{4\pi} bdb + \frac{1}{2\pi} Adb$$

- Here both sides of the duality need a spin structure (spinors in the LHS and odd Chern-Simons level in the RHS)
 - But if A is a spin_c connection,

$$\int \frac{dA}{2\pi} = \frac{1}{2} \int w_2 \text{mod } \mathbf{Z} ,$$

there is no need for a spin structure on either side.

Derive many other dualities

Starting with

$$i\bar{\Psi} \not{D}_A \Psi \quad \leftrightarrow \quad |D_b \Phi|^2 - |\Phi|^4 + \frac{1}{4\pi} bdb + \frac{1}{2\pi} Adb$$

we can derive other dualities by changing the two sides:

- Add a Chern-Simons counterterm for the classical field A
- Gauge it by turning A into a dynamical field a and adding a new classical field.
- Use other dualities.
- Repeat.

Another boson/fermion duality

For example, derive:

$$|D_B \Phi|^2 - |\Phi|^4 \quad \leftrightarrow \quad i\bar{\Psi} \not{D}_a \Psi + \frac{1}{2\pi} B da - \frac{1}{4\pi} B dB$$

LHS Wilson-Fisher fixed point

RHS QED with a single fermion, a.k.a $U(1)_{1/2}$

- Derived from the other duality
- Neither side needs a spin structure when a is a spin_c connection
- Need a Chern-Simons counterterm for B
- Can map the operators and check the phases
- RHS is T -reversal invariant (quantum symmetry)

A fermion/fermion duality

Derive: $i\bar{\Psi}\not{D}_A\Psi \leftrightarrow$

$$i\bar{\chi}\not{D}_a\chi - \frac{2}{4\pi}bdb + \frac{1}{2\pi}adb + \frac{1}{2\pi}Adb - \frac{1}{4\pi}AdA$$

LHS free fermion

RHS QED with a single fermion, coupled to $U(1)_{-2}$ of b .

- If we incorrectly integrate out b , we find the previously mentioned version with improperly quantized Chern-Simons terms.
- No need for a spin structure when a and A are spin_c connections.
- Can map the operators and check the phases
- T -reversal invariance (with anomaly) is manifest in LHS. It acts non-trivially in the RHS (quantum symmetry).

More

- ✓ Many more dualities and relations between them
- ✓ Add gravitational Chern-Simons counterterms (more checks)
- ✓ Relation to $4d$ S-duality in half-space with these $3d$ theories on the boundary (Witten's S and T operations on $3d$ field theories)
- ✓ Generalization to arbitrary N and k
 - Using a precise version of level/rank duality
 - Problem with large N_f
 - Leads to many more dualities
- Much more can be done