Parameterized post-Newtonian analysis of flyby acceleration corrections

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One natural question to ask, in connection with the flyby anomalies reported by Anderson et al. [1], is whether they can be explained by small deviations from Einstein gravity within the parameterized post-Newtonian (PPN) framework. This question has been looked at in the past by people at the Jet Propulsion Laboratory [2], but I thought it would be useful to check this, using the latest bounds on the 10 PPN parameters given in the table on page 4 of the Wikipedia article on "Parameterized post-Newtonian Formalism", and the formula for PPN modifications to a test body acceleration, given in Eqs. (6.33)-(6.34) of the book of Will [3]. The flyby velocity anomalies are of order a few to thirteen mm/s; conservatively taking them as 1 mm/s, accrued over a time window of about 10^4 seconds when the flybys cannot be tracked, one needs at least a residual acceleration of $a \sim 10^{-10}\mathrm{km/s}^2$, or converting to $c = 1$ units as used by Will, $a_{\rm Will} = a/c^2 \sim 10^{-21}\mathrm{km}^{-1}$. I went through the 11 lines in Will's formula of Eq. (6.34), and worked out an upper bound for each. Since the JPL orbital codes include the Einstein gravity contributions $\beta = 1$ and $\gamma = 1$ to the post-Newtonian acceleration, I replaced β and γ in Will's equation by $\beta - 1$ and $\gamma - 1$ to get possible corrections coming from deviations from the Einstein gravity values. As a check, I did the calculation two different ways; the more compact version is given in the scanned notes on the pages which follow. The conclusion is that all terms have an upper bound at least two orders of magnitude smaller than 10^{-21} in inverse kilometer units, and so PPN corrections to the acceleration cannot account for the flyby anomalies. Thus if corrections to gravity are invoked, one must go outside the PPN framework of theories with a symmetric metric, in which all bodies obey the equivalence principle, and in which the speed of light is constant.

References

[1] J. D. Anderson, J. K. Campbell, J. E. Ekelund, J. Ellis, and J. F. Jordan, Phys. Rev. Lett. **100**, 091102 (2008).

[2] J. K. Campbell, private communication; S. Turyshev, private communication; T. Krisher, unpublished JPL memo.

[3] C. M. Will, "Theory and Experiment in Gravitational Physics", Revised Edition, Cambridge University Press (1993), p. 149.

Check on PPN calculation

1 mm/s = 10^{-6} km/s

10^{4} s $\Rightarrow$ $\ddot{x} \sim 10^{-10}$ km/s^2

To convert to c=1 units, look at

$$\frac{\ddot{x}}{c^2} \sim 10^{-10} \frac{km}{s^2} \frac{1}{\left(3\cdot 10^5 \frac{km}{s}\right)^2} \sim 10^{-21}\ km^{-1}$$ NEEDED

In geometrized (G = c = 1) units, masses have the dimension of length and velocities are dimensionless.
So all terms in the Will formula are 1/length.

$M_\odot = 1.48 \times 10^5$ cm = 1.48 km

$M_\oplus$ = 0.44 cm = $0.44 \cdot 10^{-5}$ km

M_{NEAR} = 730 kg   so M_{NEAR} = $\frac{730}{1.99\cdot 10^{30}}$ 1.48 km = $5.4 \cdot 10^{-28}$ km

$M_\odot$ = $1.99 \cdot 10^{30}$ kg

$V_{flyby} \sim \frac{10\ km/s}{3\cdot 10^5\ km/s} \sim 3.3\cdot 10^{-5}$ ← don't use: need heliocentric velocity

V_{Earth} (in solar orbit) $\sim \frac{30\ km/s}{3\cdot 10^5\ km/s} = 10^{-4}$	$\gamma-1 < 2.3\cdot 10^{-5}$
	$\beta-1 < 2.3\cdot 10^{-4}$
V_{flyby} heliocentric < 40 km/s $\sim 1.3\cdot 10^{-4}$	$\xi < 10^{-3}$
V_{Sun} heliocentric = 0	$\alpha_1 < 10^{-4}$
$W_{Sun} = \frac{370\ km/s}{3\cdot 10^5\ km/s} \sim 1.2\cdot 10^{-3}$	$\alpha_2 < 4\cdot 10^{-7}$
$r_{flyby-earth} \sim 7000$ km	$\alpha_3 < 4\cdot 10^{-20}$
$r_{flyby-sun} \sim r_{earth-sun} \sim 1.5\cdot 10^{8}$ km	$\zeta_1 < 2\cdot 10^{-2}$
	$\zeta_2 < 4\cdot 10^{-5}$
	$\zeta_3 < 10^{-8}$
	$\zeta_4 < 6\cdot 10^{-3}$

n-body term

line 1 a = flyby b = earth

$$\frac{0.44\cdot 10^{-5}\,km}{(7\cdot 10^{3}\,km)^2}\left[4.6\cdot 10^{-5}\ \frac{0.44\cdot 10^{-5}\,km}{7\cdot 10^{3}\,km} + \underbrace{0.5\cdot 10^{-4}\ \frac{5.4\cdot 10^{-28}\,km}{7\cdot 10^{3}\,km}}_{<10^{-35}}\right]$$

$\sim 2.6\cdot 10^{-27} \ll 10^{-21}$

line 1 a = flyby b = sun

$$\frac{1.48}{(1.5\cdot 10^{8})^2}\left[4.6\cdot 10^{-5}\ \frac{1.48}{1.5\cdot 10^{8}} + \underbrace{0.5\cdot 10^{-4}\ \frac{5.4\cdot 10^{-28}}{1.5\cdot 10^{8}}}_{<10^{-40}}\right]$$

$\sim 3\cdot 10^{-29} \ll 10^{-21}$

line 2 a = flyby b = earth c = sun

$$\frac{0.44\cdot 10^{-5}}{(7\cdot 10^{3})^2}\left[4\cdot 10^{-3}\ \frac{1.48}{1.5\cdot 10^{8}}\right] \sim 3.5\cdot 10^{-24} \ll 10^{-21}$$

a = flyby b = sun c = earth

$$\frac{1.48}{(1.5\cdot 10^{8})^2}\left[2\cdot 10^{-3}\cdot \frac{0.44\cdot 10^{-5}}{7\cdot 10^{3}}\right] \sim 8\cdot 10^{-29} \ll 10^{-21}$$

Prefactors that keep occurring:

a = flyby, b = earth: $\frac{m_b}{r_{ab}^2} = \frac{0.44\cdot 10^{-5}}{(7\cdot 10^{3})^2} = 9\cdot 10^{-14}$

a = flyby, b = sun: $\frac{m_b}{r_{ab}^2} = \frac{1.48}{(1.5\cdot 10^{8})^2} = 7\cdot 10^{-17}$

line 3 a = flyby b = earth c = sun

$$9\cdot 10^{-14}\cdot\left[\underbrace{10^{-2}\ 1.48\ \frac{7\cdot 10^{3}}{(1.5\cdot 10^{8})^{2}}}_{4.6\cdot 10^{-16}} + \underbrace{10^{-3}\ \frac{1.48}{1.5\cdot 10^{8}}}_{9.8\cdot 10^{-12}}\right] \sim 9\cdot 10^{-25} \ll 10^{-21}$$

a = flyby b = sun c = earth

$$7\cdot 10^{-17}\left[\underbrace{10^{-2}\ \frac{.44\cdot 10^{-5}}{1.5\cdot 10^{8}}}_{3\cdot 10^{-16}} + \underbrace{10^{-3}\ \frac{.44\cdot 10^{-5}\ 1.5\cdot 10^{8}}{(7\cdot 10^{3})^{2}}}_{1.3\cdot 10^{-8}}\right] \sim 9\cdot 10^{-25} \ll 10^{-21}$$

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line 4 a = flyby b = earth c = sun

$$9\cdot 10^{-14}\cdot\left[\underbrace{2.3\cdot 10^{-5}\ (1.3\cdot 10^{-4})^{2}}_{3.9\cdot 10^{-13}} + \underbrace{\tfrac{1}{2}\cdot 10^{-4}\ 1.3\cdot 10^{-4}\cdot 10^{-4}}_{.7\cdot 10^{-13}} + \underbrace{1.2\cdot 10^{-5}\ (10^{-4})^{2}}_{1.2\cdot 10^{-13}}\right] \sim 5.2\cdot 10^{-26} \ll 10^{-21}$$

a = flyby b = sun c = earth

$$7\cdot 10^{-17}\left[\underbrace{2.3\cdot 10^{-5}\ (1.3\cdot 10^{-4})^{2}}_{3.9\cdot 10^{-13}} + \tfrac{1}{2}\cdot 10^{-4}\ 0 + 1.2\cdot 10^{-5}\cdot 0\right] \sim 2.7\cdot 10^{-29} \ll 10^{-21}$$

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line 5 a = flyby b = earth c = sun

$$9\cdot 10^{-14}\cdot\left[\underbrace{\tfrac{1}{2}\cdot 10^{-4}\cdot (1.2\cdot 10^{-3})^{2}}_{0.7\cdot 10^{-10} = 7\cdot 10^{-11}} + \underbrace{10^{-4}\ 1.2\cdot 10^{-3}\ 1.3\cdot 10^{-4}}_{1.6\cdot 10^{-11}}\right] \sim 7.7\cdot 10^{-24} \ll 10^{-21}$$

a = flyby b = sun c = earth

$$7\cdot 10^{-17}\cdot\left[\underbrace{\tfrac{1}{2}\cdot 10^{-4}\ (1.2\cdot 10^{-3})^{2}}_{7\cdot 10^{-11}} + \underbrace{\tfrac{1}{2}\cdot 10^{-4}\cdot 1.2\cdot 10^{-3}\ 1.3\cdot 10^{-4}}_{.8\cdot 10^{-11}}\right] \sim 5.5\cdot 10^{-27} \ll 10^{-21}$$

line 6

a = flyby b = earth c = sun

$$9\cdot10^{-19}\Big[\underbrace{\frac{4\cdot10^{-7}(10^{-4})^2}{}}_{4\cdot10^{-15}} + \underbrace{6\cdot10^{-7}(1.2\cdot10^{-3})^2}_{9\cdot10^{-13}} + \underbrace{3\cdot4\cdot10^{-7}\,1.2\cdot10^{-3}\,10^{-4}}_{1.4\cdot10^{-13}}\Big] = 9\cdot10^{-26} \ll 10^{-21}$$

a = flyby b = sun c = earth

$$7\cdot10^{-17}\Big[\underbrace{6\cdot10^{-7}\cdot(1.2\cdot10^{-3})^2}_{9\cdot10^{-13}}\Big] = 6\cdot10^{-29} \ll 10^{-21}$$

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line 7

a = flyby b = earth c = sun

$$\frac{1}{2}\,2\cdot10^{-2}\,\frac{0.44\cdot10^{-5}}{7\cdot10^{3}}\,7\cdot10^{-17} = 4\cdot10^{-28} \ll 10^{-21}$$

a = flyby b = sun c = earth

$$\frac{1}{2}\,2\cdot10^{-2}\,\overbrace{\frac{1.48}{1.5\cdot10^{8}}\,\frac{.44\cdot10^{-5}}{(1.5\cdot10^{8})^2}}^{7\cdot10^{-17}} = 2\cdot10^{-32} \ll 10^{-21}$$

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line 8

a = flyby b = earth c = sun

$$\underbrace{10^{-3}\,\frac{.44\cdot10^{-5}}{(7\cdot10^{3})^3}\,12\,1.48}_{7.4\cdot10^{-20}}\,\underbrace{|\hat{n}_{flyby-sun}-\hat{n}_{earth-sun}|}_{\sim\frac{7\cdot10^{3}}{1.5\cdot10^{8}}\sim4.7\cdot10^{-5}} \sim 3.5\cdot10^{-24} \ll 10^{-21}$$

a = flyby b = sun c = earth

$$10^{-3}\,\frac{1.48}{(1.5\cdot10^{8})^3}\,12\,.44\cdot10^{-5}\,\underbrace{|\hat{n}_{flyby-earth}-\hat{n}_{sun-earth}|}_{\lesssim2} \sim 4.6\cdot10^{-32} \ll 10^{-21}$$

line 9 $\quad a=$ flyby $\quad b=$ earth $\quad c=$ sun

$$9\cdot 10^{-14}\ 2.3\cdot 10^{-5}\ 2(1.3\cdot 10^{-4})^2 \sim 7\cdot 10^{-26} \ll 10^{-21}$$

$a=$ flyby $\quad b=$ sun $\quad c=$ earth

$$7\cdot 10^{-17}\cdot 2.3\cdot 10^{-5}(1.3\cdot 10^{-4})^2 \sim 3\cdot 10^{-29} \ll 10^{-21}$$

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line 10 $\quad a=$ flyby $\quad b=$ earth $\quad c=$ sun

$$9\cdot 10^{-14}\cdot\Big[\underbrace{10^{-4}\ 1.3\cdot 10^{-4} + 10^{-4}\ 10^{-4} + 8\cdot 10^{-7}\ 1.2\cdot 10^{-3}}_{\sim 2.3\cdot 10^{-8}}\Big]\ 10^{-4} \sim 2\cdot 10^{-25} \ll 10^{-21}$$

$a=$ flyby $\quad b=$ sun $\quad c=$ earth

$$7\cdot 10^{-17}\Big[10^{-4}\ 1.3\cdot 10^{-3} + 0 + 8\cdot 10^{-7}\ 1.2\cdot 10^{-3}\Big]\ 0 \sim 0\cdot 10^{-27} \ll 10^{-21}$$

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line 11 $\quad a=$ flyby $\quad b=$ earth $\quad c=$ sun

$$\tfrac{1}{2}\ 9\cdot 10^{-14}\Big[\underbrace{10^{-4}\ 1.3\cdot 10^{-4} + 10^{-4}\ 10^{-4} + 8\cdot 10^{-7}\ 1.2\cdot 10^{-3}}_{\sim 2.4\cdot 10^{-8}}\Big]\ 1.2\cdot 10^{-3} \sim 1.3\cdot 10^{-24} \ll 10^{-21}$$

$a=$ flyby $\quad b=$ sun $\quad c=$ earth

$$\tfrac{1}{2}\ 7\cdot 10^{-17}\Big[\underbrace{10^{-4}\ 1.3\cdot 10^{-4} + 0 + 8\cdot 10^{-7}\ 1.2\cdot 10^{-3}}_{\sim 1.4\cdot 10^{-8}}\Big]\ 1.2\cdot 10^{-3} \sim 6\cdot 10^{-28} \ll 10^{-21}$$

<u>Newtonian acceleration deviation</u> Δ = PPN correction

$$\Delta(a_\alpha^j)_{Newt} \leq m_\alpha^{-1}\left[\left(\Delta m_{I\alpha}^{jk}\right)\left|\mathcal{U}^{(0)}_{,jk}\right| + \left(m_{I\alpha}^{jk(0)}\right)\left|\Delta\mathcal{U}_{,jk}\right|\right]$$

$$|\mathcal{U}^0_{,jk}| = (9\cdot10^{-14} + 7\cdot10^{-17}) \sim 10^{-13}$$

$$m_\alpha^{-1}\,\Delta m_{I\alpha}^{jk} \sim 2\cdot10^{-2}\,\frac{\Omega_\alpha}{m_\alpha} + 3\cdot10^{-3}\,\frac{\Omega_\alpha^{lm}\cdot\hat{n}^{jklm}_\alpha}{m_\alpha} + 2\cdot10^{-2}\,\frac{\Omega_\alpha^{jk}}{m_\alpha} \sim 2\cdot10^{-26}$$

$$\frac{m^2_{Ax\,bo}}{R^3_{Ax\,by}} \quad \frac{(10^{-27})^2}{(10^{-3})^3} \quad < 10^{-46}$$

$$\frac{m_{Ax\,bo}}{R_{Ax\,by}} \quad \frac{10^{-27}}{10^{-3}} \quad 2,3\cdot10^{-26}$$

$$10^{-13} \times 2\cdot10^{-26} \sim 2\cdot10^{-39} \ll 10^{-21}$$

Even if PPN coeff are tot a 1 here, then $< 10^{-39} \ll 10^{-21}$

$m_{I\alpha}^{jk(0)} \sim 1$

$\Delta\mathcal{U}_{,jk}$ = quadrupole moment contribution

need earth + sun quadrupole moments

<u>earth</u> quadrupole moment is presumably in the JPL code

is <u>earth</u> <u>sextupole</u> in the JPL code?

$$J_{earth} \sim \frac{Q_{earth}}{m_\oplus R_\oplus^2} \sim 1.6\cdot10^{-3}$$

$$\nabla\mathcal{U}_{Earth\ quadrupole} \sim 1.6\cdot10^{-3}\,\frac{m_\oplus}{R_\oplus^2} \sim \frac{1.6\cdot10^{-3}\cdot0.44\cdot10^{-5}}{(7\cdot10^3)^2} \sim 1.4\cdot10^{-16}$$

important

So earth sextupole and octupole moments are also likely to be important - are they all in the code?

How big are they? Need $J < 10^{-8}$ to ignore!

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However earth moment forces are conservative forces which vanish asymptotically, so they won't contribute to the anomaly!

quadrupole $J_{2\,sun} \sim 2\cdot10^{-7}$ – few $\times 10^{-6}$

octupole $J_{4\,sun} \sim 10^{-6}$

∇U sun quad / octupole etc $\sim \frac{1.48}{(1.5\cdot10^8)^2}\cdot 10^{-6} \sim 6.6\cdot10^{-23} \ll 10^{-21}$ (by factor of 15)

Are solar higher moments in the code?

Any solar potential contribution to the energy change should be reduced by $\frac{140{,}000\ km}{1.5\cdot10^8\ km} \sim 10^{-3}$, since the

is the extent of $\int dr\, \nabla U$ between asymptotes

Check: $U_{sun\,in} - U_{sun\,out} \sim 10^{-3}\, U_{sun} \sim 10^{-3}\,\frac{1.48}{1.5\cdot10^8}\cdot10^{-6} \sim 10^{-17}$

$v\,\delta v \sim 10^{-6}(3\cdot10^{-4})^2 \sim 10^{-15} \sim$ flyby energy change

So potential energy change due to sun higher moments are factor 10^2 too small.

Actually it is much smaller:

$U_{sun\,quad} \sim \frac{10^{-6}}{d^3} M_\odot R_\odot^2$ $d = 1.5\cdot10^8$ km, $R_\odot = 7\cdot10^5$ km

$\sim \frac{10^{-6}}{(1.5\cdot10^8)^3}\, 1.48\,(7\cdot10^5)^2 =$ previous estimate $\cdot\left(\frac{7\cdot10^5}{1.5\cdot10^8}\right)^2 = 2\cdot10^{-5}$

So $U^{quad}_{sun\,in} - U^{quad}_{sun\,out} \sim 2\cdot10^{-22}$ as compared with

needed 10^{-15}. So uncertainties in the solar quadrupole moment cannot be an explanation.

Self-acceleration terms

$$a_{self} < 10^{-2}\left[\frac{t_A^{\,j}}{m_A} + \left(\frac{\mathcal{J}_A^{\,j}}{m_A} - \frac{3}{2}\frac{\mathcal{J}_A^{\,(k k j)}}{m_A}\right)\right] + 4\cdot10^{-5}\frac{\Omega_A^{\,j}}{m_A} + 10^{-8}\frac{\mathcal{E}_A^{\,j}}{m_A} + 2\cdot10^{-2}\frac{\mathcal{P}_A^{\,j}}{m_A}$$

$$+\; 4\cdot10^{-20}\; 1.2\cdot10^{3}\;\frac{H_A^{\,kj}}{m_A} \qquad \text{Estimate } v \le 3\,\text{cm/s} \sim 10^{-10}$$

$$\frac{\mathcal{J}_A^{\,kkj}}{m_A} \sim \frac{t_A^{\,j}}{m_A} \sim \frac{m_{Apollo}}{R_{Apollo}^{2}}\cdot\left(10^{-10}\right)^{2} \sim \frac{10^{-27}\,10^{-20}}{\left(10^{-3}\right)^{2}} \sim 10^{-41} \ll 10^{-21}$$

$$\frac{\Omega_A^{\,j}}{m_A} \sim \frac{m_{Apollo}^{2}}{R_{Apollo}^{3}} \sim \frac{\left(10^{-27}\right)^{2}}{\left(10^{-3}\right)^{3}} \sim 10^{-45} \ll 10^{-21}$$

(internal energy / unit rest mass)

$$\frac{\mathcal{E}_A^{\,j}}{m_A} \sim \frac{m_{Apollo}}{R_{Apollo}^{2}}\langle\Pi\rangle \sim \frac{10^{-27}}{\left(10^{-3}\right)^{2}}\langle\Pi\rangle \sim 10^{-21}\langle\Pi\rangle \sim 10^{-29} \ll 10^{-21}$$

$$\langle\Pi\rangle \sim \frac{10\,\text{eV}}{1\,\text{GeV}} \sim 10^{-8}$$

$$\frac{\mathcal{P}_A^{\,j}}{m_A} \sim \frac{m_{Apollo}}{R_{Apollo}^{2}}\langle p/\rho\rangle \sim 10^{-21}\langle p/\rho\rangle \ll 10^{-21}$$

$$\frac{H_A^{\,kj}}{m_A} \sim \frac{m_{Apollo}}{R_{Apollo}^{2}}\,10^{-10} \sim 10^{-31} \ll 10^{-21}$$

So the self-acceleration terms are all too small

This completes the check that all terms in

Will – GR (6.31)–(6.34)

are too small to give a 1 mm/s ΔV, by a factor of at least 10^{2}.

Can Jupiter be relevant?

Compare $\frac{m}{r}$, $\frac{m}{r^2}$, $\frac{m}{r^3}$ for earth and Jupiter

$m_{Jupiter} = 1.9 \times 10^{27}$ kg   Orbital radius of Jupiter ~ 5.25 AU
$m_{\oplus} = 6 \times 10^{24}$ kg

$m_{Saturn} = 5.7 \times 10^{26}$ kg   orbital radius of Saturn = 9.3 AU

$\frac{m_{Jupiter}}{m_{\oplus}} = 317$ $\frac{m_{Saturn}}{m_{\oplus}} = 95$

$\frac{5.3\,AU}{7000\,km} = 113{,}571$ $\frac{9.3\,AU}{7000\,km} = 199{,}286$

Jupiter $\frac{317}{113{,}571} = .003$ $\frac{317}{113{,}571^2} = 2.5 \cdot 10^{-8}$ $\frac{317}{113{,}571^3} = 2 \cdot 10^{-13}$

Saturn $\frac{95}{199{,}286} = .0005$ $\frac{9.5}{199{,}286^2} = 2 \cdot 10^{-9}$ $\frac{9.5}{199{,}286^3} = 10^{-14}$

Check on line 8 for Jupiter, since $\hat{n}$'s don't cancel for
$\hat{n}$ = flyby sun and $\hat{n}$ = Jupiter sun
a = flyby   b = Jupiter   c = sun

$\frac{0.99 \cdot 10^5\,km \cdot 317}{(5.25 \cdot 1.5 \cdot 10^8\,km)^3}$ $1.98\,km \cdot (3 \cdot 10^5\,\frac{km}{s})^2 \approx 4 \cdot 10^{-19}$ too small

So replacing earth → Jupiter just makes things smaller.