TWO WAVE PROPAGATION PROBLEMS RELATING TO AXION SEARCHES:

- VACUUM BIREFRINGENCE IN A ROTATING MAGNETIC FIELD

- DETAILED THEORETICAL ANALYSIS OF "LIGHT SHINING THROUGH A WALL" EXPERIMENTS

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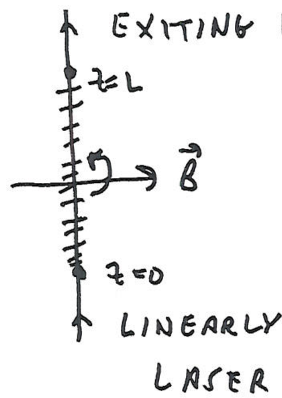
VACUUM BIREFRINGENCE IN ROTATING B FIELD

- MOTIVATION: PVLAS EXPERIMENT

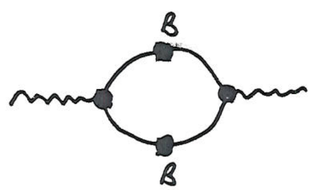
  B FIELD ON ROTATING TURNTABLE



$0 \leq z \leq L$

$\vec{B}$ FIELD ROTATING WITH ANGULAR FREQUENCY Ω AROUND BEAM AXIS

- HEISENBERG - EULER EFFECTIVE LAGRANGIAN



(+ PERMUTATIONS)

ELECTROMAGNETIC WAVE PROPAGATION IN EXTERNAL MAGNETIC FIELD

$\vec{d}$ AND $\vec{h}$ FIELDS RELATED TO

$\vec{e}$ AND $\vec{b}$ FIELDS BY $\vec{B}$ - DEPENDENT

POLARIZATION TENSORS

$$d_s = \varepsilon_{st}\, e_t \,, \quad h_s = \mu^{-1}_{st}\, b_t$$

$$\varepsilon_{st} = \delta_{st}(1-2\xi) + 7\xi\, \hat{B}_s(t)\, \hat{B}_t(t)$$

$$\mu^{-1}_{st} = \delta_{st}(1-2\xi) - 4\xi\, \hat{B}_s(t)\, \hat{B}_t(t)$$

$\hat{B}_s(t) =$ UNIT VECTOR ALONG MAGNETIC FIELD

$$\xi = \frac{\alpha^2 B^2}{45 \pi m^4}$$

B IN UNRATIONALIZED UNITS

$\alpha = 1/137.04$

$m =$ ELECTRON MASS

MAXWELL EQUATIONS ARE

$$\vec{\nabla} \cdot \vec{d} = \vec{\nabla} \cdot \vec{b} = 0$$

$$\vec{\nabla} \times \vec{e} = -\frac{\partial \vec{b}}{\partial t} \qquad \vec{\nabla} \times \vec{h} = \frac{\partial \vec{d}}{\partial t}$$

FOR TIME-INDEPENDENT $\vec{B}$,

$$n_{\vec{e} \perp \vec{B}} = (k/\omega)_\perp = 1 + 2\xi \qquad \Delta n = \frac{3}{2}\xi$$

$$n_{\vec{e} \parallel \vec{B}} = (k/\omega)_\parallel = 1 + \frac{7}{2}\xi$$

FOR $z \leq 0$ HAVE UPWARD PROPAGATING LINEARLY POLARIZED LASER BEAM

$$\vec{e} = \vec{d} = e^{i\bar{\omega}(z-t)} (\cos\theta, \sin\theta, 0)$$

$$\vec{b} = \vec{h} = e^{i\bar{\omega}(z-t)} (-\sin\theta, \cos\theta, 0)$$

IN $0 \leq z \leq L$ HAVE B FIELD PERPENDICULAR TO z AXIS, ROTATING WITH UNIFORM ANGULAR VELOCITY Ω

$$\hat{B}(t) = (\cos\Omega t, \sin\Omega t, 0)$$

$$\hat{B}'(t) = \Omega(-\sin\Omega t, \cos\Omega t, 0) = \Omega\, \hat{z} \times \hat{B}$$

PROBLEM TO SOLVE IS TO CALCULATE THE UPWARD MOVING TRANSMITTED WAVE EMERGING AT $z = L$

- BASIC STRATEGY: WAVE MATCHING

  USE ORTHONORMAL ROTATING BASIS
  $\hat{B}$ $\quad \hat{z} \times \hat{B}$

  GET PROPAGATION MODES IN ROTATING MAGNETIC FIELD "MEDIUM"

  TRANSFORM INCIDENT WAVE TO ROTATING BASIS AND MATCH INCIDENT, TRANSMITTED + REFLECTED WAVES AT $z=0$, $z=L$

  WORK TO LEADING ORDER IN SMALL QUANTITIES ξ, $\Omega/\bar{\omega}$, ΩL

  THAT IS: ξ SMALL $\quad \Omega$ SMALL

  CAN NEGLECT MULTIPLE REFLECTIONS, SO HAVE INDEPENDENT MATCHING PROBLEMS AT $z=0$, $z=L$

  ALSO, SUFFICES TO COMPUTE PHASES OF WAVES TO FIRST ORDER, BUT CORRESPONDING TRANSMITTED WAVE AMPLITUDES AT MATCHING POINTS TO ZEROTH ORDER (FIRST ORDER IN THESE WOULD GIVE L-INDEPENDENT POLARIZATION PARAMETERS)

- ALTERNATIVE STRATEGY: PERTURBATION EXPANSION IN ξ USING GREEN'S FUNCTIONS

• EM WAVE EIGENMODES IN ROTATING FIELD

WRITE

$$\vec{e} = \left[E_1 \hat{B} + E_2 \hat{z}\times\hat{B}\right] e^{ikz - i\omega t}$$

$$\vec{b} = \left[B_1 \hat{B} + B_2 \hat{z}\times\hat{B}\right] e^{ikz - i\omega t}$$

FROM EQS. FOR $\vec{d}$, $\vec{h}$ AND MAXWELL EQS., GET COUPLED EQS. FOR $E_{1,2}$ AND $B_{1,2}$

$$k E_1 = \omega B_2 + \Omega i B_1$$

$$k i E_2 = -\omega i B_1 - \Omega B_2$$

$$k i B_1 = -\omega (1 + 4\xi) i E_2 + \Omega E_1$$

$$k B_2 = \omega (1 + 7\xi) E_1 - \Omega i E_2$$

det = 0 ⇒ DISPERSION RELATION

$$\left[k^2 - \omega^2 (1+7\xi)\right]\left[k^2 - \omega^2 (1+4\xi)\right] - 2\Omega^2(\omega^2 + k^2) + \Omega^4 = 0$$

SOLVE TO GIVE SOLUTIONS $k_\pm$ UPWARD MOVING

$-k_\pm$ DOWNWARD MOVING

AND RATIOS $\frac{B_2}{E_1}$, $\frac{E_2}{E_1}$, $\frac{B_1}{E_1}$ FOR EACH MODE

- WAVE MATCHING

$z=0$ INCIDENT WAVE TO ROTATING FIELD MODES

$z=L$ ROTATING FIELD MODES TO EXITING WAVE

AFTER MUCH ALGEBRA, GET A SIMPLE FINAL ANSWER

$$\vec{e}\Big|_{z=L} = e^{i\bar{\omega}(L-t)}(X, Y, 0)$$

$$\vec{b}\Big|_{z=L} = e^{i\bar{\omega}(L-t)}(-Y, X, 0)$$

$$X = \cos\theta\left(1 + \frac{11}{4} i\bar{\omega}L\xi\right) + \frac{3}{4} i\bar{\omega}L\xi \cos(2\Omega t - \theta)$$

$$Y = \sin\theta\left(1 + \frac{11}{4} i\bar{\omega}L\xi\right) + \frac{3}{4} i\bar{\omega}L\xi \sin(2\Omega t - \theta)$$

- POLARIZATION PARAMETERS OF EXITING WAVE

FOLLOW BORN + WOLF EXPOSITION

WRITE

$$X = d_1 e^{i\delta_1}$$

$$Y = d_2 e^{i\delta_2}$$

$$d_1 = \cos\theta \qquad d_2 = \sin\theta$$

$$\delta_{\{\substack{1\\2}} = \frac{11}{4}\,\bar{\omega} L \xi + \frac{3}{4}\,\bar{\omega} L \xi \begin{cases} \cos(2\Omega t - \theta)/\cos\theta \\ \sin(2\Omega t - \theta)/\sin\theta \end{cases}$$

BORN-WOLF ANGLE α GIVEN BY

$$\tan\alpha = \frac{d_2}{d_1} = \tan\theta$$

$\Rightarrow$ TO FIRST ORDER, NO ROTATION OF POLARIZATION AXIS OF BEAM

BORN-WOLF ANGLE χ GIVEN BY

$$\sin 2\chi = \sin 2\alpha \, \sin(\delta_2 - \delta_1) \qquad \Rightarrow$$

$$\Psi = \text{ELLIPTICITY} = \tan\chi \approx \chi \approx \tfrac{1}{2}(\delta_2 - \delta_1)\sin 2\theta$$

$$= -\frac{3}{4}\,\xi\,\bar{\omega} L \sin 2(\theta - \Omega t)$$

USING $\bar{\omega} = \frac{2\pi}{\lambda}$ $\quad$ $\xi = \frac{2\,\Delta n}{3}$

THIS IS

$$\Phi = -\frac{\pi L}{\lambda}\,\Delta n\,\sin 2(\theta - \Omega t)$$

SO TO FIRST ORDER, ELLIPTICITY IS WHAT ONE CALCULATES ASSUMING THE $\vec{B}$ FIELD TO BE FROZEN AT ITS INSTANTANEOUS POSITION DURING TRAVERSAL BY THE LASER BEAM, USING THE FORMULA FOR ELLIPTICITY CALCULATED FROM VACUUM BIREFRINGENCE IN A STATIC MAGNETIC FIELD - THIS CONFIRMS THE PVLAS GROUP CALCULATION (See also Biswas + Melnikov arXiv: hep-ph /0611345)

WAVE MATCHING, AFTER MUCH ALGEBRA, GIVES A VERY SIMPLE ANSWER - SUGGESTS THERE SHOULD BE ANOTHER WAY TO SOLVE THE PROBLEM

- PERTURBATION METHOD (APPLIES TO A z AND t DEPENDENT MAGNETIC FIELD)

SUBSTITUTE GENERAL ANSATZ

$$\vec{e} = E_1(z,t)\,\hat{B} + E_2(z,t)\,\hat{z}\times\hat{B}$$

$$\vec{b} = B_1(z,t)\,\hat{B} + B_2(z,t)\,\hat{z}\times\hat{B}$$

AND MAKE A PERTURBATIVE EXPANSION IN $\xi(z,t)$:

$$E_{1,2} = E_{1,2}^{(0)} + E_{1,2}^{(1)} + \cdots$$

$$B_{1,2} = B_{1,2}^{(0)} + B_{1,2}^{(1)} + \cdots$$

ORDER ξ

INCIDENT WAVE ON ROTATING BASIS

$$E_1^{(0)} = e^{i\bar{\omega}(z-t)}\cos(\theta - \Omega t) \qquad \text{etc}$$

GET INHOMOGENEOUS WAVE EQUATIONS FOR $E_{1,2}^{(1)}$:

$$\left(\partial_z^2 - \partial_t^2\right) E_{1,2}^{(1)} = I_{1,2}$$

$$I_1 = 7\,\xi\,\partial_t^2 E_1^{(0)} + 12\,\partial_t\xi\,\partial_t E_1^{(0)} + 5\,\partial_t^2\xi\,E_1^{(0)} - 2\,\partial_z\xi\,\partial_t E_1^{(0)} - 2\,\partial_z\partial_t\xi\,E_1^{(0)}$$

$$I_2 = 4\,\xi\,\partial_t^2 E_2^{(0)} + 2\,\partial_t\xi\,\partial_t E_2^{(0)} - 2\,\partial_t^2\xi\,E_2^{(0)} - 6\,\partial_z\xi\,\partial_t E_2^{(0)} - 6\,\partial_z\partial_t\xi\,E_2^{(0)}$$

SOLVE USING GREEN'S FUNCTION $G(z,t)$ FOR 1-DIMENSIONAL WAVE EQUATION WITH OUTGOING WAVE BOUNDARY CONDITION

$$G(z,t) = -\frac{1}{2}\,\theta(t - |z|)$$

$$(\partial_z^2 - \partial_t^2)\, G(z,t) = \delta(z)\,\delta(t)$$

$\Rightarrow$

$$E^{(1)}_{1,2}(z,t) = \int_0^L dz' \int_{-\infty}^{\infty} dt'\; G(z-z', t-t')\; I_{1,2}(z',t')$$

WHEN MAGNETIC FIELD IS UNIFORM $0 \le z \le L$ AND t-INDEPENDENT, HAVE

$$\partial_z \xi = \bar{\xi}\,[\,\delta(z) - \delta(z-L)\,]$$

$$I_1 = e^{i\bar{\omega}(z-t)} \cos(\theta - \Omega t)\,[-7\xi\bar{\omega}^2 + 2i\bar{\omega}\,\partial_z \xi]$$

$$I_2 = e^{i\bar{\omega}(z-t)} \sin(\theta - \Omega t)\,[-4\xi\bar{\omega}^2 + 6i\bar{\omega}\,\partial_z \xi]$$

EVALUATING THE INTEGRAL AT $z=L$ GET

$$E_1|_{z=L} = e^{i\bar{\omega}(z-t)} \cos(\theta - \Omega t)\left(1 + \frac{7}{2}\, i\bar{\omega}\xi L\right)$$

$$E_2|_{z=L} = e^{i\bar{\omega}(z-t)} \sin(\theta - \Omega t)\left(1 + \frac{4}{2}\, i\bar{\omega}\xi L\right)$$

TRANSFORMING TO FIXED BASES GET

$$e^{i\bar{\omega}(z-t)}\,(X, Y)$$

SAME X,Y AS BEFORE BY WAVE-MATCHING

- PVLAS NUMBERS $\Rightarrow$

$\Omega L = 0.7 \times 10^{-8}$

$2\Omega/\bar{\omega} = 2 \times 10^{-15}$ ALL VERY SMALL

$3\xi/2 = 10^{-22}$

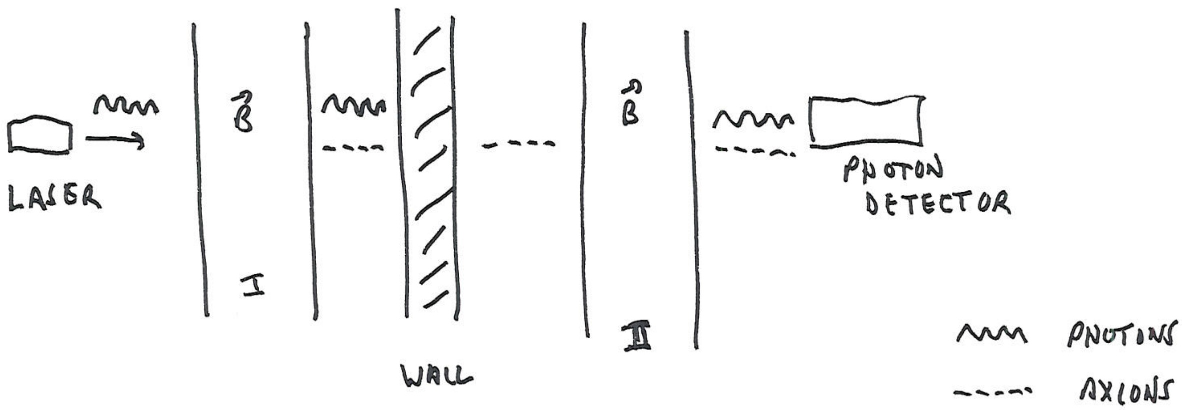
HOWEVER, PROPAGATION MODES IN $\vec{B}$ REGION INVOLVE THE SQUARE ROOT

$$\sqrt{\left(\frac{2\Omega}{\bar{\omega}}\right)^2 + \left(\frac{3\xi}{2}\right)^2}$$

SO PVLAS IS IN THE <u>LARGE</u> - Ω REGIME OF THE $\sqrt{}$ AND THE EIGENMODES IN ROTATING FIELD REGION

HOWEVER, ALL DEPENDENCE ON THE $\sqrt{}$ CANCELS IN THE TRANSMITTED WAVE PARAMATERS - DEPENDENCE OF POLARIZATION + ELLIPTICITY ON ξ, Ω IS SMOOTH AROUND ORIGIN IN ξ, Ω PARAMETER PLANE

DETAILED THEORETICAL ANALYSIS OF "LIGHT SHINING THROUGH A WALL" EXPERIMENTS



$\vec{B}$ REGION I CONVERTS PHOTONS TO AXIONS

WALL ABSORBS PHOTONS ; AXIONS GO THROUGH

$\vec{B}$ REGION II CONVERTS AXIONS BACK TO PHOTONS

- THE PROBLEM: GET FORMULAS FOR
  PHOTON → AXION
  AXION → PHOTON
  CONVERSION , ASSUMING NON-ZERO AXION MASS m

- ACTION S

$\frac{1}{2}\epsilon_{\mu\nu\alpha\beta}F^{\alpha\beta}$

$$S=\int d^4x\left[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}-\frac{1}{2}\phi(\partial^2+m^2)\phi+\frac{g}{4}\phi F^{\mu\nu}\tilde{F}_{\mu\nu}\right]$$

FREE PHOTON FREE AXION AXION-PHOTON COUPLING

TAKE $\hat{B}$ FIELD IN x DIRECTION

AXIS OF PROPAGATION IS z DIRECTION

WRITE $\qquad \beta = g B \qquad E_x = \partial_t a$

THEN ACTION IS

$$S = -\frac{1}{2}\int d^4x \left[a\,\partial_z^2 a + \phi(\partial_z^2 + m^2)\phi - 2\beta\phi\,\partial_t a \right]$$

$\Rightarrow$ EQUATIONS OF MOTION

$$(\partial_t^2 - \partial_z^2 + m^2)\,\phi(z,t) = \beta(z)\,\partial_t a(z,t)$$

$$(\partial_t^2 - \partial_z^2)\,a(z,t) = -\beta(z)\,\partial_t \phi(z,t)$$

EVERYTHING FOLLOWS FROM THESE:

- ϕ, a, $\partial_z \phi$, $\partial_z a$ CONTINUOUS ACROSS BOUNDARIES

- CURRENT CONSERVATION

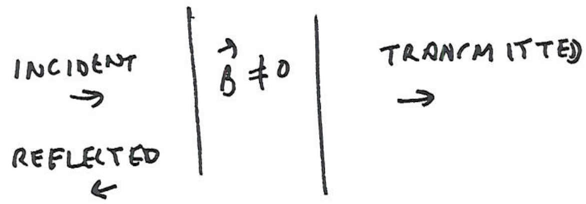
$$\partial_t J_0 + \partial_z J_z = 0$$

$$J_0 = \phi^* \partial_t \phi - \partial_t \phi^* \phi + a^* \partial_t a - \partial_t a^* a + \beta(z)(a^*\phi - \phi^* a)$$

$$J_z = \partial_z \phi^* \phi - \phi^* \partial_z \phi + \partial_z a^* a - a^* \partial_z a$$

IN $\vec{B} = 0$ REGIONS,

$$a \propto e^{\pm i\omega z} \qquad \phi \propto e^{\pm ikz} \qquad k = \sqrt{\omega^2 - m^2}$$



$$\phi(z,t) = e^{-i\omega t}\left(\phi_I e^{ikz} + \phi_R e^{-ikz}\right) \qquad \phi(z,t) = e^{-i\omega t}\,\phi_T\, e^{ikz}$$

$$a(z,t) = e^{-i\omega t}\left(a_I e^{i\omega z} + a_R e^{-i\omega z}\right) \qquad a(z,t) = e^{-i\omega t}\, a_T\, e^{i\omega z}$$

CURRENT CONSERVATION $\Rightarrow$ UNITARITY RELATION BETWEEN INCIDENT, REFLECTED, TRANSMITTED WAVE AMPLITUDES

$$k\,|\phi_I|^2 + \omega\,|a_I|^2 = k\left(|\phi_R|^2 + |\phi_T|^2\right) + \omega\left(|a_R|^2 + |a_T|^2\right)$$

NOW WANT TO CALCULATE ϕ_T, a_T

FOR $\phi_I = 0$, $a_I = 1$, DESCRIBING A PHOTON BEAM INCIDENT FROM LEFT - USE

- GREEN'S FUNCTION METHOD (SIMPLER)
- WAVE MATCHING METHOD (GIVES ALL ORDERS)

- GREEN'S FUNCTION CALCULATION

WORK TO FIRST ORDER IN $\beta(z)$

EXPAND

$$\phi(z,t) = \phi^{(0)}(z,t) + \phi^{(1)}(z,t) + \cdots$$
$$a(z,t) = a^{(0)}(z,t) + a^{(1)}(z,t) + \cdots$$

ZEROTH ORDER:

$$(\partial_t^2 - \partial_z^2 + m^2)\, \phi^{(0)}(z,t) = 0$$

$$(\partial_t^2 - \partial_z^2)\quad a^{(0)}(z,t) = 0$$

FIRST ORDER:

$$(\partial_t^2 - \partial_z^2 + m^2)\, \phi^{(1)}(z,t) = \beta(z)\, \partial_t\, a^{(0)}(z,t)$$

$$(\partial_t^2 - \partial_z^2)\quad a^{(1)}(z,t) = -\beta(z)\, \partial_t\, \phi^{(0)}(z,t)$$

NEED $\quad G_m(z,t) \quad$ OBEYING

$$(\partial_t^2 - \partial_z^2 + m^2)\, G_m(z,t) = \delta(z)\,\delta(t)$$

FOURIER SPACE:

$$G_m(z,t) = -\frac{1}{(2\pi)^2}\int dk\, d\omega\, \frac{e^{i(kz-\omega t)}}{(\omega-\omega_k)(\omega+\omega_k)}$$

$\omega_k = \sqrt{k^2+m^2}$

$\mp\omega_k \to \mp\omega_k - i\epsilon \quad$ GIVES RETARDED SOLUTION

CLOSING ω CONTOUR UP/DOWN GIVES

$$G_m^R(z,t) = -\theta(t)\,\Delta_m(z,t)$$

$$\Delta_m(z,t) = \frac{i}{4\pi}\int \frac{dk}{\omega_k}\left(e^{i(kz+\omega_k t)} - e^{i(kz-\omega_k t)}\right)$$

$$= \int \frac{dk}{2\pi}\,\frac{\sin(k|z| - \omega_k t)}{\omega_k}$$

FOR UNIT AMPLITUDE INCIDENT PHOTON,

$$a^{(0)} = e^{i\bar{\omega}(z-t)} \qquad \phi^{(0)} = 0$$

USING GREEN'S FUNCTION,

$$\phi^{(1)}(z,t) = -i\bar{\omega}\int_0^L dz'\,\beta(z')\,e^{i\bar{\omega}z'}\int_{-\infty}^{\infty} dt'\,G_m^R(z-z', t-t')\,e^{-i\bar{\omega}t'}$$

DOING THE t' INTEGRAL WE HAVE

$$\phi^{(1)}(z,t) = -i\bar{\omega}\,e^{-i\bar{\omega}t}\int_0^L dz'\,\beta(z')\,e^{i\bar{\omega}z'}$$

$$\times \int_{-\infty}^{\infty}\frac{dk}{2\pi}\,\frac{e^{ik(z-z')}}{\left(k-\sqrt{\bar{\omega}^2-m^2}\right)\left(k+\sqrt{\bar{\omega}^2-m^2}\right)}$$

TO GET $a^{(0)} = 0$ IN INFINITE PAST,

SET $\bar{\omega} \to \bar{\omega} + i\varepsilon$ $\quad \varepsilon > 0$ THIS MOVES

THE POLES OFF THE REAL AXIS

FOR $z - z' > 0$ CAN CLOSE k CONTOUR UP
$z - z' < 0$ " " " DOWN

SO GET TWO PIECES: THE TRANSMITTED AND REFLECTED PARTS

$$\phi^{(1)}(z,t) = \phi^{(1)}_{Trans}(z,t) + \phi^{(1)}_{Reflect}(z,t)$$

$$\phi^{(1)}_{Trans}(z,t) = \frac{\bar{\omega}}{2\sqrt{\bar{\omega}^2 - m^2}} e^{-i\bar{\omega}t + iz\sqrt{\bar{\omega}^2 - m^2}} \times \int_0^z dz' \beta(z') e^{iz'\left(\bar{\omega} - \sqrt{\bar{\omega}^2 - m^2}\right)}$$

$$\phi^{(1)}_{Reflect}(z,t) = \frac{\bar{\omega}}{2\sqrt{\bar{\omega}^2 - m^2}} e^{-i\bar{\omega}t - iz\sqrt{\bar{\omega}^2 - m^2}} \times \int_z^L dz' \beta(z') e^{iz'\left(\bar{\omega} + \sqrt{\bar{\omega}^2 - m^2}\right)}$$

NOW TAKE $\beta(z) = \begin{cases} \beta = \text{CONSTANT} & 0 \le z \le L \\ 0 & \text{OUTSIDE THIS INTERVAL} \end{cases}$

FIND FOR $z \ge L$:

$$\phi^{(1)}_{Trans}(z,t) = \frac{\beta\bar{\omega}\, e^{i\left(z\sqrt{\bar{\omega}^2 - m^2} - \bar{\omega}t\right)}}{m^2\sqrt{\bar{\omega}^2 - m^2}} e^{i\frac{L}{2}\left(\bar{\omega} - \sqrt{\bar{\omega}^2 - m^2}\right)} \times \left(\bar{\omega} + \sqrt{\bar{\omega}^2 - m^2}\right) \sin\left(\frac{L}{2}\left(\bar{\omega} - \sqrt{\bar{\omega}^2 - m^2}\right)\right)$$

THIS HAS MODULUS SQUARED

$$|\phi^{(1)}_{Tran}(z,t)|^2 = \frac{\beta^2}{m^4}\left(\frac{\bar{\omega}^2}{\bar{\omega}^2 - m^2}\right)\left(\bar{\omega} + \sqrt{\bar{\omega}^2 - m^2}\right)^2$$

$$\times \sin^2\left(\frac{L}{2}\left(\bar{\omega} - \sqrt{\bar{\omega}^2 - m^2}\right)\right)$$

UNITARITY $\Rightarrow$ $1 \geq \frac{k}{\omega}|\phi_T|^2$

OR IN PRESENT NOTATION

$$1 \geq \underbrace{\frac{\sqrt{\bar{\omega}^2 - m^2}}{\bar{\omega}}\,|\phi^{(1)}(z \geq L, t)|^2}_{\approx \frac{\beta^2}{m\sqrt{\bar{\omega}^2 - m^2}} \sin^2\left(\frac{mL}{2}\right) \text{ FOR } \bar{\omega} \approx m}$$

$$\to \infty \quad \text{AS} \quad \bar{\omega} \to m\,!$$

SO THERE IS A THRESHOLD CUSP FOR AXION MASS $m \neq 0$. WILL HAVE TO DO AN ALL ORDERS CALCULATION TO SEE HOW UNITARITY IS RESTORED (THE ∞ IS JUST AN ARTIFACT OF WORKING ONLY TO FIRST ORDER)

RESULT IS DIFFERENT FOR A UNIT INCIDENT AXION WAVE WITH NO INCIDENT PHOTON: FIND FOR TRANSMITTED PHOTON AT $z \geq L$

$$a^{(1)}_{\text{Trans}}(z,t) = -\frac{e^{i\bar{\omega}(z-t)}}{2} \int_0^L dz' \, \beta(z') \, e^{iz'\left(\sqrt{\bar{\omega}^2-m^2}-\bar{\omega}\right)}$$

NO CUSP (OR DIP) AT $\bar{\omega} \approx m$

- ALL ORDERS CALCULATION - SKETCH

B FIELD IN $0 \leq z \leq L$

$z \leq 0$
$$\phi(z) = \phi_I e^{ikz} + \phi_R e^{-ikz}$$
$$a(z) = a_I e^{i\omega z} + a_R e^{-i\omega z}$$

$z \geq L$
$$\phi(z) = \phi_T e^{ikz} + \tilde{\phi} e^{-ikz}$$
$$a(z) = a_T e^{i\omega z} + \tilde{a} e^{-i\omega z}$$

LATER IN CALCULATION, WILL IMPOSE CONDITION $\tilde{\phi} = \tilde{a} = 0$

IN MAGNETIC FIELD REGION, GET

FOUR VALUES $k = \pm k_+ \quad \pm k_-$

$$(k_\pm)^2 = \omega^2 - \frac{m^2}{2}\left(1 \mp \sqrt{1+x^2}\right) \qquad x = \frac{2\beta\omega}{m^2}$$

WRITE $\delta = \dfrac{-ix}{1+\sqrt{1+x^2}}$ FIELDS ARE

$$\phi(z) = \delta A_0^+ e^{izk_+} + \Phi_0^- e^{izk_-} + \delta a_0^+ e^{-izk_+} + \varphi_0^- e^{-izk_-}$$

$$a(z) = A_0^+ e^{izk_+} + \delta \Phi_0^- e^{izk_-} + a_0^+ e^{-izk_+} + \delta \varphi_0^- e^{-izk_-}$$

NOW MATCH AT $z = 0$ $\Rightarrow$

$$\begin{pmatrix} \phi_I \\ \phi_R \\ a_I \\ a_R \end{pmatrix} = M_{12} \begin{pmatrix} \Phi_0^- \\ \varphi_0^- \\ A_0^+ \\ a_0^+ \end{pmatrix}$$

MATCH AT $z = L$ $\Rightarrow$

$$\begin{pmatrix} \Phi_0^- \\ \varphi_0^- \\ A_0^+ \\ a_0^+ \end{pmatrix} = M_{34} \begin{pmatrix} \phi_T \\ \tilde{\phi} \\ a_T \\ \tilde{a} \end{pmatrix}$$

COMBINING

$$\begin{pmatrix} \phi_I \\ \phi_R \\ d_I \\ d_R \end{pmatrix} = S \begin{pmatrix} \phi_T \\ \tilde{\phi} \\ d_T \\ \tilde{a} \end{pmatrix} \qquad S \equiv M_{12} M_{34}$$

S MATRIX ELEMENTS ARE TABULATED IN PAPER ; MANY SYMMETRIES GIVE INTERNAL CHECKS ON THE ALGEBRA

CASE OF INCIDENT PHOTON FROM LEFT:

$$\phi_I = 0 \qquad \tilde{a} = \tilde{\phi} = 0$$

$$\Rightarrow \quad 0 = S_{11} \phi_T + S_{13} d_T$$

$$\phi_R = S_{21} \phi_T + S_{23} d_T$$

$$1 = S_{31} \phi_T + S_{33} d_T$$

$$d_R = S_{41} \phi_T + S_{43} d_T$$

EASILY SOLVED TO GIVE THE TRANSMITTED AND REFLECTED AMPLITUDES:

$$\phi_T = \frac{S_{13}}{D} \qquad d_T = -\frac{S_{11}}{D}$$

$$\phi_R = \frac{S_{21}S_{13} - S_{23}S_{11}}{D} \qquad d_R = \frac{S_{41}S_{13} - S_{43}S_{11}}{D}$$

$$D = S_{13}S_{31} - S_{11}S_{33}$$

- LEADING ORDER IN $X = 2\omega\beta/m^2$

  RECOVER GREEN'S FUNCTION EXPRESSIONS

- ω NEAR THRESHOLD

  FIND CUSP SHAPE NEAR $\omega \approx m$

$$|\phi_T|^2 \simeq \frac{\sin^2(\frac{1}{2}mL)}{\left(\frac{|\beta|}{m}\sin^2(\frac{1}{2}mL) + \frac{k}{|\beta|}\right)^2 + \frac{|\beta|^2}{4m^2}\left(\sin(mL) - mL\right)^2}$$

$$\frac{k}{m}|\phi_T|^2 \le \frac{\frac{k}{m}\sin^2(\frac{1}{2}mL)}{\left(\frac{|\beta|}{m}\sin^2(\frac{1}{2}mL) + \frac{k}{|\beta|}\right)^2} \le \frac{1}{4}$$

USING (FOR $s, t > 0$) $\quad \dfrac{st}{(s+t)^2} = \dfrac{1}{4}\,\dfrac{(s+t)^2 - (s-t)^2}{(s+t)^2} \le \dfrac{1}{4}$

WHEN $mL \gg 1$, $\sin^2(\frac{1}{2} mL)$ HAS TO BE EVALUATED AS AN AVERAGE OF

$$\sin^2\left(\tfrac{1}{2}L\left(\omega-\sqrt{\omega^2-m^2}\right)\right) \approx \sin^2\left(\tfrac{1}{2}L\left(m-\sqrt{2m(\omega-m)}\right)\right)$$

WHEN THERE ARE MANY OSCILLATIONS IN AN INTERVAL OF WIDTH Δ NEAR THRESHOLD,

$$\langle \sin^2 \cdots \rangle \to 1/2$$

$$\langle \sin^4 \cdots \rangle \to 3/8$$

AVERAGE PHOTON TRANSMISSION PROBABILITY IN "LIGHT SHINING THROUGH A WALL" IS, IN INTERVAL $m \le \omega \le m+\Delta$ NEAR THRESHOLD,

$$\bar{P} \cong \frac{1}{\Delta}\int_m^{m+\Delta} d\omega\, P(\omega) \simeq \frac{\beta^4}{2\Delta m^3} \underbrace{\langle \sin^4 \cdots \rangle}_{\approx \frac{3}{8}} \log\left(\frac{2m^3\Delta}{\beta^4}\right)$$

FROM LOWER CUTOFF PROVIDED BY ALL ORDERS CALCULATION

<u>NO</u> INCREASE WITH L!

(SINCE $\frac{\beta^4}{\Delta m^3} \ll 1$, log IS POSITIVE; AS $\Delta \to 0$, HAVE TO USE OSCILLATORY FORMULA FOR $\sin^4 \cdots$)

- IS THE CUSP USEFUL?

NOT CLEAR RECALL

$$|\varphi^{(1)}_{\text{Trans}}|^2 = \frac{\beta^2}{m^4}\left(\frac{\bar{\omega}^2}{\bar{\omega}^2 - m^2}\right)\left(\bar{\omega} + \sqrt{\bar{\omega}^2 - m^2}\right)^2$$

$$\times \sin^2\left(\frac{L}{2} \frac{m^2}{\bar{\omega} + \sqrt{\bar{\omega}^2 - m^2}}\right)$$

$$\underset{\substack{m \to 0 \\ \bar{\omega} \text{ NOT NEAR } 0}}{\longrightarrow} \quad \frac{\beta^2}{m^4} \frac{L^2}{4} m^2 = \frac{1}{4}\beta^2 L^2$$

$\bar{P} \approx \frac{1}{16}\beta^4 L^4$ AWAY FROM CUSP

$\bar{P} \approx \frac{3}{16}\beta^4 \frac{1}{\Delta m^3} \log\left(\frac{2m^3\Delta}{\beta^4}\right)$ CUSP

SO TRADEOFF COMPARISON IS

$$L^4 \leftrightarrow \frac{3}{\Delta m^3} \log\left(\frac{2m^3\Delta}{\beta^4}\right)$$

FOR NARROW ENOUGH LASER BANDWIDTHS, CUSP COULD BE USEFUL