AST 513: Stellar Systems

Assignment 4

Due Wednesday, November 28, 2007

1. Disk potentials

(a) What is the potential in the $z = 0$ plane corresponding to a razor-thin surface-density distribution of the form $\Sigma(R) = \Sigma_0(R_0/R)^\alpha$? Be sure to state the range of α for which your formula is valid.

(b) What is the potential in the $z = 0$ plane corresponding to a razor-thin surface-density distribution of the form $\Sigma(R) = \Sigma_0 \exp(-r^2/2a^2)$?

2. The Sun's orbit

Using the epicycle approximation, find the minimum and maximum distances from the Galactic center that the Sun attains in its orbit. You may assume that the present solar distance is 8 kpc, Oort's constants are $A = 15\,\mathrm{km\,s^{-1}\,kpc^{-1}}$, $B = -12\,\mathrm{km\,s^{-1}\,kpc^{-1}}$, and the present solar velocity relative to the local circular speed (*not* relative to the guiding center of the epicycle) is $-10.0\,\mathrm{km\,s^{-1}}$ in the radial direction and $5.2\,\mathrm{km\,s^{-1}}$ in the direction of rotation (i.e., the Sun is moving towards the Galactic center and faster than a circular orbit).

3. The velocity field near the Sun

Let $\hat{\mathbf{e}}_1$ and $\hat{\mathbf{e}}_2$ be unit vectors in an inertial coordinate system centered on the Sun, with $\hat{\mathbf{e}}_1$ pointing away from the Galactic center (towards $\ell = 180°$, $b = 0$) and $\hat{\mathbf{e}}_2$ pointing towards $\ell = 270°$, $b = 90°$. The mean velocity field $\mathbf{v}(\mathbf{x})$ relative to the Local Standard of Rest can be expanded in a Taylor series,

$$v_i = \sum_{j=1}^{2} H_{ij} x_j + \vee(x^2).$$

(a) Assuming that the Galaxy is stationary and axisymmetric, evaluate the matrix $\mathbf{H}$ in terms of the Oort constants A and B.

(b) What is the matrix $\mathbf{H}$ in a rotating frame, that is, if $\hat{\mathbf{e}}_1$ continues to point to the center of the Galaxy as the Sun orbits around it?

(c) In a homogeneous, isotropic universe, there is an analogous 3×3 matrix $\mathbf{H}$ that describes the relative velocity $\mathbf{v}$ between two fundamental observers separated by $\mathbf{x}$. Evaluate this matrix in terms of the Hubble constant.

4. Stability of circular orbits

Prove that circular orbits in a given potential are unstable if the angular momentum per unit mass on a circular orbit decreases outward.

5. Relation of action to potential

A particle moves in a one-dimensional potential well $\Phi(x)$. In angle-action variables, the Hamiltonian has the form $H(J) = cJ^{4/3}$ where c is a constant. Find $\Phi(x)$.

6. The standard map

Surfaces of section for Hamiltonian systems with two degrees of freedom can be regarded as area-preserving maps in two dimensions, and many of the properties of surfaces of section can therefore be studied through maps rather than by integrating equations of motion. The maps are numerically much faster, easier, and more accurate.

The standard or Chirikov-Taylor map is a map of the plane (w, I), given by

$$I_{n+1} = I_n + K \sin w_n, \qquad w_{n+1} = w_n + I_{n+1}.$$

Here I_n and w_n are both defined modulo 2π. The parameter K is known as the stochasticity parameter.

(a) Show that the standard map preserves area in the (w, I) plane.

(b) Plot surfaces of section for $K = 0.3, 1, 2$. Note that the fraction of the area occupied by chaotic orbits increases with K. Note: plotting informative surfaces of section requires a good choice of the number, initial conditions, and duration of each orbit—too few and the structure of the surface of section is not clearly delineated; too many and the image is confusing. In contrast to other questions in the assignment, the grade for this one will depend on the beauty of the result!