Think Globally, Act Locally

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Quantum Fields beyond Perturbation Theory, KITP 2014

Ofer Aharony, NS, Yuji Tachikawa, arXiv:1305.0318
Anton Kapustin, Ryan Thorngren, arXiv:1308.2926, arXiv:1309.4721
Anton Kapustin, NS, arXiv:1401.0740

Seemingly unrelated questions

1. *2d* Ising has two phases:
   1. High T, unbroken global $\mathbb{Z}_n$
   2. Low T, broken global $\mathbb{Z}_n$, n vacua

   Duality should exchange them. How can this be consistent?

   Similar question for the *4d* $\mathbb{Z}_n$ lattice gauge theory
2. In $su(n)$ gauge theory 't Hooft and Wilson operators satisfy the equal time ('t Hooft) commutation relations:

$$W(\gamma)\mathcal{T}(\gamma') = e^{2\pi i\ell(\gamma,\gamma')/n}\mathcal{T}(\gamma')W(\gamma)$$

They are spacelike separated, so how can they fail to commute?

Seemingly unrelated questions

3. The de-confinement transition of a *4d* $SU(n)$ gauge theory is related to a global $\mathbb{Z}_n$ symmetry in *3d*. What is its *4d* origin? Can we gauge it?
4. The partition function $\mathcal{Z} = \sum c_k \mathcal{Z}_k$ is a sum of contributions from distinct topological sectors (e.g. twisted boundary conditions in orbifolds, different instanton numbers in a gauge theory).
   1. What are the physical restrictions on c_k?
   2. Are there new theories with different values of c_k (e.g. restrict the sum over the instanton number)?
5. S-duality in $\mathcal{N} = 4$ SUSY maps $SU(2) \leftrightarrow SO(3)$. How is this consistent with $(ST)^3 = 1$?
6. Can we relate $SU(n)$ and $SU(n)/\mathbb{Z}_n$ gauge theories as in orbifolds?

Coupling a QFT to a TQFT

Unified framework:

Couple an ordinary quantum field theory to a topological theory.

In many cases such a coupling affects the local structure, e.g.:

- Free matter fields coupled to a Chern-Simons theory in *3d*.
- Orbifolds in *2d* CFT

Often the local structure is not affected, but there are still interesting consequences: spectrum of line and surface operators, local structure after compactification…

Outline

- Line operators
- Higher form global symmetries and their gauging
- Review of a simple TFT – a *4d* $\mathbb{Z}_n$ gauge theory
- $SU(n)/\mathbb{Z}_n$ gauge theories
- Modifying the sum over topological sectors (constraining the instanton number – restricting the range of θ)
- Topological $\mathbb{Z}_n$ lattice gauge theory
- Turning an $SU(n)$ lattice gauge theory to an $SU(n)/\mathbb{Z}_n$ theory
- Duality in *2d* Ising and *4d* $\mathbb{Z}_n$ lattice gauge theory

- Answering the seemingly unrelated questions

Line operators

Some line operators are boundaries of surfaces.

1. If the results depend on the geometry of the surface, this is not a line operator.
2. In some cases the dependence on the surface is only through its topology.
3. Genuine line operators are independent of the surface.

$$W(\gamma)\mathcal{T}(\gamma') = e^{2\pi i \ell(\gamma,\gamma')/n}\mathcal{T}(\gamma')W(\gamma)$$

Here at least one of the line operators needs a surface. Hence, the apparent lack of locality.

Genuine line operators of the form $W(\gamma)^n\mathcal{T}(\gamma)^m$ with appropriate n and m are relatively local.

Higher-form global symmetries

Continuous q-form global symmetry – transformation with a closed q-form $\epsilon^{(q)}$ ($q = 0$ is an ordinary global symmetry with constant ϵ).

Discrete q-form global symmetry $\int \epsilon^{(q)} \in 2\pi\mathbb{Z}$.

Example:

An ordinary gauge theory with group G is characterized by transition functions $g_{ij} \in G$ with $g_{ij}g_{jk}g_{ki} = 1$.

If no matter fields transforming under C , the center of G, 1-form discrete global symmetry $g_{ij} \to h_{ij}g_{ij}$ with $h_{ij} \in$ C and $h_{ij}h_{jk}h_{ki} = 1$.

Higher-form global symmetries

1-form discrete global symmetry $g_{ij} \to h_{ij} g_{ij}$ with $h_{ij} \in \mathrm{C}$ and $h_{ij} h_{jk} h_{ki} = 1$.

When compactified on a circle, this 1-form global symmetry leads to an ordinary global symmetry C.

It is common in thermal physics – the Polyakov loop is the order parameter for its breaking.

We gauge C by relaxing $h_{ij} h_{jk} h_{ki} = 1$ (analog of gauging an ordinary global symmetry by letting ϵ depend on position).

The resulting theory is an ordinary gauge theory of G/C.

A simple TFT – a *4d* $\mathbb{Z}_n$ gauge theory

[Maldacena, Moore, NS; Banks, NS]

1. Can describe as a $\mathbb{Z}_n$ gauge theory.
2. Can introduce a compact scalar $\phi \sim \phi + 2\pi$ and a $U(1)$ gauge symmetry $\phi \to \phi + n\,\lambda$ (with $\lambda \sim \lambda + 2\,\pi$).
3. Can also introduce a $U(1)$ gauge field A with Lagrangian

$$\frac{i}{2\pi} H \wedge (d\phi + nA)$$

   H is a 3-form Lagrange multiplier. $U(1) \to \mathbb{Z}_n$ manifest.

4. Can dualize ϕ to find

$$\frac{in}{2\pi} B \wedge F$$

   with $F = dA$ and $H = dB$.

The basic TFT – a *4d* $\mathbb{Z}_n$ gauge theory

$$\frac{in}{2\pi} B \wedge F$$

5. Can dualize A to find

$$\frac{i}{2\pi} F \wedge (d\hat{A} + nB)$$

F is a 2-form Lagrange multiplier.

Gauge symmetry:

$$\hat{A} \to \hat{A} + d\hat{\lambda} - n\Lambda$$

$$B \to B + d\Lambda$$

6. Can keep only $\hat{A}$ with its gauge symmetry
7. Locally, can fix the gauge $\hat{A} = 0$ and have only a $\mathbb{Z}_n$ 1-form gauge symmetry

Observables in a *4d* $\mathbb{Z}_n$ gauge theory

$$\frac{in}{2\pi} B \wedge F$$

Two kinds of Wilson operators

$$W_A(\gamma) = e^{i \oint_\gamma A}$$

$$W_B(\Sigma) = e^{i \oint_\Sigma B}$$

Their correlation functions

$$\langle W_B(\Sigma) W_A(\gamma) \rangle = e^{2\pi i \ell(\Sigma, \gamma)/n}$$

No additional ('t Hooft) operators using $\hat{A}$ or ϕ – they are trivial.

An added term in a *4d* $\mathbb{Z}_n$ gauge theory

$$\frac{i}{2\pi}F\wedge(d\hat{A}+nB) \qquad\qquad \frac{in}{2\pi}B\wedge F$$

In any of the formulations we can add the term [Gukov, Kapustin; Kapustin, Thorngren]

$$\frac{ipn}{4\pi}B\wedge B=\frac{ip}{4\pi n}d\hat{A}\wedge d\hat{A}$$

With the modified gauge transformations:

$$\hat{A}\to\hat{A}+d\hat{\lambda}-n\Lambda$$

$$B\to B+d\Lambda$$

$$A\to A-p\Lambda$$

Consistency demands $\frac{pn}{2}\in\mathbb{Z}$ and $p\sim p+2n$.

Variants of the *4d* $\mathbb{Z}_n$ gauge theory

An obvious generalization is to use q and $(D-q-1)$-form gauge fields in D dimensions

$$\frac{in}{2\pi} A^{(q)} \wedge dA^{(D-q-1)}$$

This is particularly interesting for $q = 0$ (or $q = D - 1$)

$$\frac{in}{2\pi} \Phi \wedge dA^{(D-1)} = \frac{in}{2\pi} \Phi \wedge F^{(D)}$$

Low energy theory of a system with a global $\mathbb{Z}_n$ symmetry.

The order parameter for the symmetry breaking is $e^{i\Phi}$.

The Wilson operator

$$W_A(\Sigma) = \exp\left(i \oint_\Sigma A^{(D-1)}\right)$$

describes a domain wall between different vacua.

From $SU(n)$ to $SU(n)/\mathbb{Z}_n$ gauge theories

$G = SU(n)$ gauge theory

- The Wilson line W is a genuine line operator
- The 't Hooft line $\mathcal{T}$ needs a surface (the Dirac string). Hence the nonlocality in the commutation relations

$$W(\gamma)\mathcal{T}(\gamma') = e^{2\pi i \ell(\gamma,\gamma')/n}\mathcal{T}(\gamma')W(\gamma)$$

- If no matter fields charged under the $\mathbb{Z}_n$ center
  - Only the topology of the surface is important. (Like disorder operator in the Ising model with vanishing magnetic field.)
  - Global 1-form discrete symmetry C= $\mathbb{Z}_n$

From $SU(n)$ to $SU(n)/\mathbb{Z}_n$ gauge theories

Next, we gauge the 1-form symmetry C= $\mathbb{Z}_n$ to find an $SU(n)/\mathbb{Z}_n$ g.t.

Can use any of the formulations of a C= $\mathbb{Z}_n$ gauge theory.

Extend $SU(n)$ to $U(n)$ by adding $\hat{A}$ and impose the 1-form gauge symmetry $\hat{A} \to \hat{A} - n\,\Lambda$ to remove the added local dof.

We can also add a new term in this theory – a discrete θ-term

$$\frac{ip}{4\pi n} d\hat{A} \wedge d\hat{A} \qquad \frac{pn}{2} \in \mathbb{Z} \qquad p \sim p + 2n$$

(Interpreted as an $SU(n)/\mathbb{Z}_n$ theory, it is identified with the Pontryagin square ${w_2}^2$ of the gauge bundle [Aharony, NS, Tachikawa]. Here, a manifestly local expression for it.)

$SU(n)/\mathbb{Z}_n$ gauge theory – operators

Use e.g.

$$\frac{i}{2\pi} F \wedge (d\hat{A} + nB) + \frac{ipn}{4\pi} B \wedge B \qquad \frac{pn}{2} \in \mathbb{Z}$$

$$B \to B + d\Lambda$$

$$\hat{A} \to \hat{A} - n\Lambda$$

$$F \to F - pd\Lambda$$

The surface operator

$$e^{i \oint_\Sigma B} = e^{-\frac{i}{n} \oint_\Sigma d\hat{A}}$$

measures the ‘t Hooft magnetic flux (w_2 of the bundle) through Σ.

It is a manifestly local expression – integral of a local density.

(More complicated expression for torsion cycles.)

$SU(n)/\mathbb{Z}_n$ gauge theory

Wilson $W(\gamma) = \left(W_f^{SU(n)}(\gamma) e^{\frac{i}{n}\oint_\gamma \hat{A}} \right) e^{i\int_\Sigma B}$ $\quad \partial\Sigma = \gamma$

't Hooft $\mathcal{T}(\gamma) = e^{i\oint_\gamma A + ip\int_\Sigma B}$

The dependence on Σ is topological.

Genuine line operators (dyonic) $\quad \mathcal{T}(\gamma)W(\gamma)^{-p}$

The parameter p is a discrete θ-parameter [Aharony, NS, Tachikawa].

- It labels distinct $SU(n)/\mathbb{Z}_n$ theories.
- It can be understood either as a new term in the Lagrangian $\frac{i\,p}{4\pi n} d\,\hat{A} \wedge d\,\hat{A}$ (it is the Pontryagin square w_2^2 of the gauge bundle), or in terms of the choice of genuine line operators.

Restricting the range of the θ-angle [NS 2010]

- Similarly, we can restrict the range of θ by coupling a standard gauge theory to a $\mathbb{Z}_n$ topological gauge theory (the one related to a broken global $\mathbb{Z}_n$ symmetry)

$$\frac{in}{2\pi}\Phi F^{(4)} \quad \text{with} \quad \Phi \sim \Phi + 2\pi \quad ; \quad \int F^{(4)} \in 2\pi\mathbb{Z}$$

$$\cdots + \frac{i\theta}{16\pi^2}\mathrm{Tr}F\tilde{F} + \frac{in}{2\pi}\Phi F^{(4)} + \frac{i\Phi}{16\pi^2}\mathrm{Tr}F\tilde{F}$$

- The integral over Φ forces the topological charge to be a multiple of n. *H*ence, $\theta \sim \theta + 2\pi/n$.
- θ and $\theta + 2\pi/n$ are in the same superselection sector.
- Φ is a "discrete axion."
- Note, this is consistent with locality and clustering!

Lattice gauge theory

- The variables of a lattice gauge theory are group elements on the links U_l. The gauge symmetry acts on the sites and the gauge invariant interaction is in terms of products around the plaquettes $U_p = \mathrm{Tr}\,(\Pi_l\, U_l)$.
- For a $\mathbb{Z}_n$ gauge theory we write $U_l = \exp(2\pi i\, u_l/n)$ and $U_p = \exp(2\pi i\, u_p/n)$.
- A 1-form gauge symmetry (Kalb-Ramond) resides on the links with gauge fields on the plaquettes. Such a $\mathbb{Z}_n$ gauge theory has variables $V_p = \exp(2\pi i\, v_p/n)$ and the gauge invariant variables on the cubes are $V_c = \exp(2\pi i\, v_c/n)$.

Topological gauge theory on the lattice

A topological $\mathbb{Z}_n$ gauge theory is based on $U_l = \exp(2\pi i\, u_l/n)$ on the links and the Boltzmann weight

$$\prod_p U_p^{v_p} = \prod_p e^{2\pi i u_p v_p/n}$$

with u_p are derived from u_l.

$V_p = \exp(2\pi i\, v_p/n) \in \mathbb{Z}_n$ are gauge fields on the dual of p.

They impose the constraint $U_p = \exp(2\pi i\, u_p/n) = 1$.

- This theory differs from the ordinary $\mathbb{Z}_n$ lattice gauge theory by this flatness constraint.
- Note the similarity to BF-theories.
- Easy to generalize to higher form gauge theories.

From $SU(n)$ to $SU(n)/\mathbb{Z}_n$ on the lattice

Starting with an $SU(n)$ lattice gauge theory with $U_l \in SU(n)$ construct an $SU(n)/\mathbb{Z}_n$ gauge theory [Halliday, Schwimmer].

An $SU(n)$ lattice gauge theory has a global 1-form symmetry $U_l \to h_l U_l$ with $h_l \in \mathbb{Z}_n$ and $h_p = \Pi_l\, h_l = 1$.

We gauge it by relaxing the constraint $h_p = 1$.

As with the continuum presentation of the $\mathbb{Z}_n$ theory above, there are several ways to do it:

- Make the Boltzmann weight an invariant function, e.g. a function of $|\mathrm{Tr}\, U_p|^2$, or
- Add a $\mathbb{Z}_n$ gauge field B_p on the plaquettes…

From $SU(n)$ to $SU(n)/\mathbb{Z}_n$ on the lattice

- Add a $\mathbb{Z}_n$ gauge field B_p on the plaquettes. In order not to add new dof, add an integer modulo n Lagrange multiplier v_c on the cubes. The Boltzmann weight is

$$\left(\prod_c B_c^{v_c}\right) \prod_p f(B_p \mathrm{Tr}\ U_p)$$

More precisely, $V_c = \exp\left(2\pi i\ v_c/n\right) \in \mathbb{Z}_n$ are gauge fields on the dual of the cubes.

The first factor is a $\mathbb{Z}_n$ topological gauge theory.

Can also add the discrete θ-parameter on the lattice…

From $SU(n)$ to $SU(n)/\mathbb{Z}_n$ on the lattice

The $SU(n)$ Wilson loop $\mathrm{Tr}(U_l\, U_l \ldots U_l)$ is not invariant under the 1-form symmetry. But

$$W_U = \mathrm{Tr}(U_l U_l \ \ldots U_l) B_p B_p \ldots B_p$$

is gauge invariant – we tile it with a topological surface. It can still detect confinement.

The 't Hooft operator is

$$\mathcal{T} = V_c V_c \ldots V_c$$

with the product over all the cubes penetrated by a curve. The presence of $\mathcal{T}$ changes the constraint on B_c in those cubes.

Closed surface operators $W_B = B_p B_p \ldots B_p$ measures the 't Hooft magnetic flux w_2.

Duality in the *2d* Ising model

- The dynamical variables are $\mathbb{Z}_n$ spins S_s on the sites and the Boltzmann weight is $\prod_{s,\ell} f\left(S_s \bar{S}_{s+\ell}\right)$.
- After duality the Boltzmann weight is

$$\left(\prod_{p^*} S_{p^*}^{v_{p^*}}\right) \prod_{\ell^*} \tilde{f}\left(\tilde{S}_{s^*} V_{\ell^*} \bar{\tilde{S}}_{s^*+\ell^*}\right)$$

  - The variables are on the dual lattice.
  - The first factor is a topological $\mathbb{Z}_n$ gauge theory; the $\mathbb{Z}_n$ gauge field V_{ℓ^*} is flat.
  - Locally, pick the gauge $V_{\ell^*} = 1$ to find another Ising system.
  - Globally, we need to keep the topological sector.
- The $2d$ Ising model is dual to the $2d$ Ising model coupled to a $\mathbb{Z}_n$ topological gauge theory – an orbifold of Ising.

Duality in $\mathbb{Z}_n$ lattice gauge theory

- It is often stated that the *3d* $\mathbb{Z}_n$ lattice gauge theory is dual to the Ising model and the *4d* $\mathbb{Z}_n$ lattice gauge theory is selfdual.
- More precisely, we need to couple them to a $\mathbb{Z}_n$ topological lattice gauge theory:
  - In *3d* it is an ordinary (0-form) $\mathbb{Z}_n$ gauge theory
  - In *4d* it is a 1-form $\mathbb{Z}_n$ topological gauge theory – variables on the plaquettes and the cubes are constrained to be 1.
- In *4d* $\mathbb{Z}_n$ lattice gauge theory
  - At strong coupling – confinement
  - At weak coupling – the ordinary $\mathbb{Z}_n$ gauge symmetry is unbroken – a topological phase, not Higgs.
  - Duality exchanges Higgs and confinement. But since this system is not quite selfdual, there is no contradiction.

Answers to the seemingly unrelated questions

1. *2d* Ising is not selfdual. It is dual to an orbifold of Ising – Ising coupled to a topological $\mathbb{Z}_n$ gauge theory.
2. The spacelike commutation relations

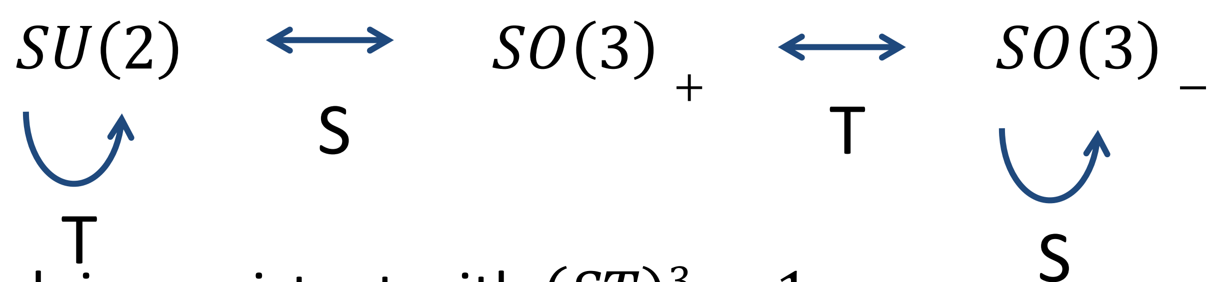
$$W(\gamma)\mathcal{T}(\gamma') = e^{2\pi i\ell(\gamma,\gamma')/n}\mathcal{T}(\gamma')W(\gamma)$$

   are consistent because at least one of these operators includes a (topological) surface.
3. The global $\mathbb{Z}_n$ symmetry of a *4d* $SU(n)$ gauge theory on a circle originates from a global 1-form symmetry in *4d*. Gauging it has the effect of changing the *4d* gauge group to $SU(n)/\mathbb{Z}_n$.

Answers to the seemingly unrelated questions

4. Not all consistency conditions on c_k in $\mathcal{Z} = \sum c_k \mathcal{Z}_k$ are understood. For a given gauge group there are several distinct consistent options, including changing the sum over instanton sectors. They can be described by coupling the system to a TQFT.
5. Depending on p there are two (actually, 4) distinct $SO(3)$ gauge theories in *4d*, $SO(3)_\pm$ [Gaiotto, Moore, Neitzke].

In $\mathcal{N} = 4$ SUSY [Aharony, NS, Tachikawa]



which is consistent with $(ST)^3 = 1$.

Answers to the seemingly unrelated questions

6. Unified descriptions of orbifolds and *4d* gauge theories
   - In orbifolds we start with a system with a global symmetry.
     - Background gauge field – twisted boundary conditions
     - Gauging the symmetry by summing over these sectors
     - This removes operators and includes others
     - Discrete torsion: different coefficients for the sectors
   - In *4d* gauge theories the global symmetry is a 1-form symmetry
     - Twisted sectors are bundles of a quotient of the gauge group
     - Gauging the 1-form global symmetry – summing over sectors
     - This changes the line and surface operators
     - Discrete θ-parameter – different coefficients for the sectors

Conclusions

It is interesting to couple an ordinary QFT to a TQFT.

- The resulting theory can have a different local structure.
- More generally, it has different line and surface operators.
- When placed on manifolds other than $\mathbb{R}^D$ the effects are often more dramatic.
- Such a coupling to a TQFT can describe the difference between a theory with gauge group G and a theory with gauge group G/C, e.g. $SU(n)$ and $SU(n)/\mathbb{Z}_n$.
- It also allows us to describe additional coupling constants like discrete θ-parameters, or restrictions on the range of the ordinary θ-angle in a manifestly local way.
- Such added TQFT also resolve problems with duality (*2d* Ising, *4d* lattice gauge theories, $\mathcal{N} = 1, 4$ theories in *3d* and *4d*).

Thank you for your attention