ISIMA lectures on celestial mechanics. 3

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1. The stability of planetary systems

To understand the formation and evolution of exoplanet systems, we would like to have empirical theoretical tools for predicting whether a given planetary system is stable—or, more precisely, what is its lifetime before some catastrophic event such as a collision or ejection—without having to integrate the planetary orbits for the lifetime of the host star. There are two discouraging lessons from studies of the stability of the solar system that we should bear in mind. First, small changes in the initial conditions or system parameters can lead to large changes in the lifetime. Second, chaos does not necessarily imply instability: in the solar system the likely lifetime is at least 1000 times longer than the Liapunov time.

One approach to this problem is to consider planetary systems in which both the planet mass and the separation between adjacent planets are very small. In examining this limit a useful parameter is the **Hill radius**. Consider a planet of mass m in a circular orbit of radius r around a host star of mass M. A test particle also circles the host star, in an orbit of radius $r + x$ with $0 < x \ll r$. The planet orbits with angular speed $n = (GM/r^3)^{1/2}$ which is slightly faster than the angular speed of the test particle, $[GM/(r + x)^3]^{1/2}$. As the planet overtakes the test particle the maximum force per unit mass that it exerts on the test particle is $F \sim Gm/x^2$ and this force lasts for a duration $\Delta t \sim n^{-1}$; thus the velocity change is $\sim F\Delta t \sim Gm/(x^2 n)$. The change is as large as the initial relative velocity $v \sim nx$ if $F\Delta t \gtrsim v$ which implies $x^3 \lesssim Gm/n^2 = r^3(m/M)$, or $x \lesssim r_H$ where

$$r_H = r \left(\frac{m}{3M}\right)^{1/3} \tag{1}$$

is the Hill radius. The Hill radius appears in other contexts in astrophysics, such as the tidal or limiting radius of star clusters, the Roche limit for solid satellites, and the location of the Lagrange points (the factor of 3 in this formula ensures that the Hill radius is equal to the distance to the collinear Lagrange points).

For two planets of comparable mass, $m_1, m_2 \ll M$, the analog is the **mutual Hill radius**, defined as

$$r_H = \frac{r_1 + r_2}{2} \left(\frac{m_1 + m_2}{3M}\right)^{1/3}. \tag{2}$$

The use of the arithmetic mean of the radii (rather than, say, the geometric mean) is arbitrary but the choice of what mean to use makes very little difference since the mutual Hill radius is only dynamically important when r_1 and r_2 are similar, since we have assumed that $m_1, m_2 \ll M$. In contrast, we use the sum of the masses (rather than, say, $(m_1/3M)^{1/3} + (m_2/3M)^{1/3}$) because the dynamics of two small bodies during a close encounter depends on the masses only through their sum (Petit & Hénon 1986, Icarus 66, 536).

A system of two small planets on nearby circular, coplanar orbits is stable for all time if $|r_1 - r_2| > 3.46 r_H$ (the standard reference is Gladman 1993, Icarus 106, 247, although forms of this criterion were derived earlier). Note that the scaling $|r_1 - r_2| > \text{const} \times (m/M)^{1/3}$ does not capture all of the interesting dynamics of two bodies on nearby orbits; for example, the criterion that the orbits are regular rather than chaotic is $|r_1 - r_2| > \text{const} \times (m/M)^{2/7}$ (Wisdom 1980, AJ 85, 1122).

For systems of three or more planets there are no exact stability criteria. Moreover such systems are not stable for all time—the instability time becomes very large if the planets have small masses or large separations but such systems always eventually exhibit instability. Numerical orbit integrations imply that a system of three or more planets on nearly circular, coplanar orbits is stable for at least N orbital periods if the separation between adjacent planets is $|r_{i+1} - r_i| > K(N) r_H$ where, for example, $K(10^{10}) \simeq 9$–12 (e.g., Smith & Lissauer 2009, Icarus 201, 381).

In summary, a system of planets on nearly circular, coplanar orbits is expected to be stable for the stellar lifetime if the separation of adjacent planets exceeds ~ 10 mutual Hill radii. To do better than this crude criterion, our only option is to integrate the planetary orbits.

2. Hamiltonian perturbation theory

2.1. The disturbing function

The dynamics of any one body in the N-body problem can be described by a Hamiltonian $H(\mathbf{q}, \mathbf{p}) = H_K(\mathbf{q}, \mathbf{p}) + H_1(\mathbf{q}, \mathbf{p}, t)$ where $H_K = \frac{1}{2}|\mathbf{p}|^2 - \mathfrak{G} M/|\mathbf{q}|$ is the Kepler Hamiltonian and H_1 represents the perturbing gravitational potential from the other planets, or from passing stars, the equatorial bulge of the host star, etc. We are interested mainly in near-Keplerian problems where $|H_1| \ll |H_K|$. The goal of perturbation theory in celestial mechanics is to find solutions for motion in this Hamiltonian that are valid when H_1 is "small enough"—in fact it is for this problem that perturbation theory was invented. The goal of this section is to persuade you that Hamiltonian perturbation theory is straightforward to apply in simple

systems, and provides answers and insight that cannot easily be obtained in any other way.

In a Hamiltonian system we can replace $(\mathbf{q}, \mathbf{p})$ with any other canonical coordinates and momenta. In particular we may use the Delaunay elements[1] to obtain

$$H = -\frac{(\mathfrak{G} M)^2}{2L^2} + H_1(L, G, H, \ell, \omega, \Omega, t). \tag{3}$$

The function H_1 is called the **disturbing function** (unfortunately in many celestial mechanics books this name is used to refer to $-H_1$). Deriving H_1 is much harder in Delaunay elements than in Cartesian coordinates (for example, it depends on all six Delaunay elements but only three Cartesian coordinates) but the advantage of working in Delaunay elements is so great that this effort is usually worthwhile.

Some restrictions on the form of H_1 can be derived from general considerations. If we replace ℓ by $\ell + 2\pi$ the position of the particle is unchanged, so H_1 is unchanged. Thus H_1 must be a periodic function of ℓ with period 2π. Similarly it must be a periodic function of ω and Ω with the same period. Thus it can be written as a Fourier series

$$\begin{aligned} H_1 &= \sum_{k,m,n} H_{kmn}(L, G, H, t) \cos[k\ell + m\omega + n\Omega + \phi_{\ell mn}(t)] \\ &= \sum_{k,m,n} H_{kmn}(L, G, H, t) \cos[k\lambda + (m-k)\varpi + (n-m)\Omega + \phi_{kmn}(t)], \end{aligned} \tag{4}$$

where as usual the mean longitude $\lambda = \ell + \omega + \Omega$ and the longitude of periapsis $\varpi = \omega + \Omega$. When the eccentricity is zero ($G = L$), H_1 cannot depend on the periapsis direction ϖ. Similarly, when the inclination is zero ($H = G$), H_1 cannot depend on the nodal longitude Ω. More generally, H_1 has to be an analytic function of the position and this restricts the possible forms of H_1 to satisfy the **d'Alembert property**[2]

$$H_{kmn}(L, G, H, t) = \mathrm{O}[(L - G)^{|m-k|/2}, (G - H)^{|n-m|/2}]. \tag{5}$$

Now suppose that the perturbing potential is stationary in a rotating frame, $\Phi[R, z, \phi - \phi_s(t)] = \sum_{m'} \Phi_{m'}(R, z) \cos m'[\phi - \phi_s(t)]$. If we rotate the origin of the azimuthal coordinate system by ψ then ϕ and ϕ_s both increase by ψ but the potential remains the same. Similarly,

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[1] As a reminder, the Delaunay momenta are $L = (GMa)^{1/2}$, $G = L(1 - e^2)^{1/2}$, $H = G \cos i$ and the conjugate coordinates are the mean anomaly ℓ, the argument of periapsis ω, and the longitude of the ascending node Ω.

[2] Similarly, if a smooth function $f(x, y)$ of the Cartesian coordinates x and y is rewritten in polar coordinates (r, ϕ) then its expansion in powers of r will have the form $c_0 + c_1 r \cos(\phi - \phi_1) + c_2 r^2 \cos 2(\phi - \phi_2) + \cdots$.

the longitude of the node Ω increases by ψ but the elements ℓ and ω do not change. In order that H_1 does not change, we then require that $m = m'$ and $\phi_{kmn} = -m\phi_s(t)$.

Thousands of pages have been written about the disturbing function in the gravitational N-body problem and its approximate forms for nearly circular, nearly coplanar orbits like those found in the solar system. Here we will discuss only three specific problems to give some flavor of these results and how the disturbing function is used in practice.

2.2. Satellite orbits

We first analyze some of the properties of the orbit of a satellite around a planet. The external gravitational potential from a planet with an axis of symmetry can be written

$$\Phi(\mathbf{r}) = -\frac{\mathfrak{G}\,m}{r}\left[1 - \sum_{n=2}^{\infty}\frac{J_n R^n}{r^n}P_n(\cos\theta)\right] \tag{6}$$

where m and R are the mass and mean radius of the planet, θ is the polar angle from the symmetry axis, P_n is a Legendre polynomial, and J_n is a dimensionless multipole moment. There is no $n = 1$ component if the center of mass is chosen to be the origin of the coordinates. In most cases the dominant component is $n = 2$; since $P_2(\cos\theta) = \frac{3}{2}\cos^2\theta - \frac{1}{2}$ we then have

$$\Phi(\mathbf{r}) = -\frac{\mathfrak{G}\,m}{r} + \frac{\mathfrak{G}\,mJ_2R^2}{r^5}(3z^2 - r^2) \tag{7}$$

where z is the component of the radius vector along the symmetry axis. The corresponding disturbing function is

$$H_1 = \frac{\mathfrak{G}\,mJ_2R^2}{2r^5}(3z^2 - r^2). \tag{8}$$

In converting this potential to a disturbing function of the form (4) we note that Φ has no explicit time dependence so ϕ_{kmn} is independent of time. Since the mean longitude λ rotates through 2π in one orbital period, while ϖ and Ω are constants for a Keplerian orbit, the terms in the disturbing function are of two types: **short-period** terms with $k \neq 0$ and **secular** terms with $k = 0$. Although the short-period terms are important for precise predictions of the location of the satellite at a given instant, the largest changes in its orbit will come from the secular terms since these act for a much longer time.

Restricting the disturbing function to its secular terms is equivalent to averaging over one orbit. Denoting this average by $\langle\cdot\rangle$ and evaluating $\langle z^2/r^5\rangle$ and $\langle 1/r^3\rangle$ using equations (50) and (43) of Lecture 1, we find

$$H_{\text{sec}} = \langle H_1\rangle = \frac{\mathfrak{G}\,mJ_2R^2}{4a^3(1-e^2)^{3/2}}(3\sin^2 I - 2) = \frac{(\mathfrak{G}\,m)^4 J_2R^2}{4L^3G^3}(1 - 3H^2/G^2). \tag{9}$$

With this disturbing function, most of the properties of the satellite orbit are easy to understand. For example, Hamilton's equations imply that $dL/dt = -\partial H_{\text{sec}}/\partial \ell = 0$ so the semi-major axis $a = L^2/(\mathfrak{G}M)$ is constant, except for small short-period variations. Similarly, $dG/dt = -\partial H_{\text{sec}}/\partial \omega = 0$ so the angular momentum and eccentricity are constants except for short-period variations[3]. Similarly $dH/dt = -\partial H_{\text{sec}}/\partial \Omega = 0$ so H is constant and the inclination $I = \cos^{-1} H/G$ is fixed. The apsis direction or perigee ω precesses at a rate

$$\begin{aligned}
\dot{\omega} &= \frac{\partial H_{\text{sec}}}{\partial G} = \frac{3(\mathfrak{G}m)^4 J_2 R^2}{4L^3 G^6}(5H^2 - G^2) \\
&= \frac{3(\mathfrak{G}m)^4 J_2 R^2}{4L^3 G^6}(5H^2 - G^2) \\
&= \frac{3(\mathfrak{G}m)^{1/2} J_2 R^2}{4a^{7/2}(1-e^2)^2}(5\cos^2 I - 1).
\end{aligned} \tag{10}$$

Thus the apsis direction of low-inclination orbits precesses forward; the apsis direction of high-inclination orbits precesses backward; and at the so-called **critical inclination** $I = \cos^{-1}\sqrt{1/5} = 63.4°$ the apsis does not precess.

The node precesses at a rate

$$\dot{\Omega} = \frac{\partial H_{\text{sec}}}{\partial H} = -\frac{3(\mathfrak{G}m)^{1/2} J_2 R^2}{2a^{7/2}(1-e^2)^2}\cos I, \tag{11}$$

which is retrograde for all orbits with $0 \le I < 90°$.

2.3. Lidov-Kozai oscillations

Suppose that a planet of mass m orbits a host star of mass m_0, with $m + m_0 = M$. Its position relative to the host star is $\mathbf{x}$. There is a companion star of mass m_c with position $\mathbf{x}_c$ relative to the host star, with $|\mathbf{x}_c| \gg |\mathbf{x}|$. Expanding the heliocentric potential (eq. 37 of Lecture 2) in inverse powers of $|\mathbf{x}_c|$,

$$\begin{aligned}
\Phi(\mathbf{x}) &= -\frac{\mathfrak{G}M}{x} - \frac{\mathfrak{G}m_c}{x_c\sqrt{1 - 2\mathbf{x}\cdot\mathbf{x}_c/x_c^2 + x^2/x_c^2}} + \frac{\mathfrak{G}m_c\mathbf{x}\cdot\mathbf{x_c}}{x_c^3} \\
&= -\frac{\mathfrak{G}M}{x} - \frac{\mathfrak{G}m_c}{x_c^3}\left[x_c^2 + \mathbf{x}\cdot\mathbf{x}_c - \tfrac{1}{2}x^2 + \frac{3(\mathbf{x}\cdot\mathbf{x_c})^2}{2x_c^2}\right] + \frac{\mathfrak{G}m_c\mathbf{x}\cdot\mathbf{x_c}}{x_c^3} + \mathrm{O}(x^3/x_c^4) \\
&= -\frac{\mathfrak{G}M}{x} + \frac{\mathfrak{G}m_c}{x_c^3}\left[\tfrac{1}{2}x^2 - \frac{3(\mathbf{x}\cdot\mathbf{x_c})^2}{2x_c^2}\right] + \mathrm{O}(x^3/x_c^4);
\end{aligned} \tag{12}$$

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[3]This is a property of the quadrupole potential that is not shared by other multipoles.

in the last line we have dropped an unimportant constant term $-\mathfrak{G}\, m_c/x_c$. Dropping the terms of order x^3/x_c^4 gives us the **quadrupole approximation**, in which the disturbing function is

$$H_1 = \frac{\mathfrak{G}\, m_c}{x_c^3} \left[\tfrac{1}{2}x^2 - \frac{3(\mathbf{x} \cdot \mathbf{x_c})^2}{2x_c^2} \right]. \tag{13}$$

Let $\hat{\mathbf{e}}$, $\hat{\mathbf{u}}$, $\hat{\mathbf{n}}$ be unit vectors pointing respectively towards periapsis ($f = 0$), towards $f = \tfrac{1}{2}\pi$ in the orbital plane, and along the angular-momentum vector (thus $\hat{\mathbf{e}} \times \hat{\mathbf{u}} = \hat{\mathbf{n}}$). Then $\mathbf{x} = r(f)(\cos f\,\hat{\mathbf{e}} + \sin f\,\hat{\mathbf{u}})$ with a similar formula for $\mathbf{x_c}$. As in the previous section, the largest effect on the orbit comes from the secular terms, which are obtained by averaging H_1 over the orbit. In this case we want to average over *both* the inner and outer orbits:

$$H_{\text{sec}} = \langle H_1 \rangle = \mathfrak{G}\, m_c \left[\tfrac{1}{2}\langle r^2\rangle \left\langle \frac{1}{r_c^3} \right\rangle - \tfrac{3}{2}(\hat{\mathbf{e}} \cdot \hat{\mathbf{e}}_c)^2 \langle r^2\cos^2 f\rangle \left\langle \frac{\cos^2 f_c}{r_c^3} \right\rangle \right.$$

$$- \tfrac{3}{2}(\hat{\mathbf{e}} \cdot \hat{\mathbf{u}}_c)^2 \langle r^2\cos^2 f\rangle \left\langle \frac{\sin^2 f_c}{r_c^3} \right\rangle - \tfrac{3}{2}(\hat{\mathbf{u}} \cdot \hat{\mathbf{e}}_c)^2 \langle r^2\sin^2 f\rangle \left\langle \frac{\cos^2 f_c}{r_c^3} \right\rangle$$

$$\left. - \tfrac{3}{2}(\hat{\mathbf{u}} \cdot \hat{\mathbf{u}}_c)^2 \langle r^2\sin^2 f\rangle \left\langle \frac{\sin^2 f_c}{r_c^3} \right\rangle \right]. \tag{14}$$

Using equations (43) of Lecture 1, this simplifies to

$$H_{\text{sec}} = \frac{\mathfrak{G}\, m_c a^2}{4a_c^3(1 - e_c^2)^{3/2}} \left\{ 2 + 3e^2 - \tfrac{3}{2}[(\hat{\mathbf{e}} \cdot \hat{\mathbf{e}}_c)^2 + (\hat{\mathbf{e}} \cdot \hat{\mathbf{u}}_c)^2](1 + 4e^2) \right.$$

$$\left. - \tfrac{3}{2}[(\hat{\mathbf{u}} \cdot \hat{\mathbf{e}}_c)^2 + (\hat{\mathbf{u}} \cdot \hat{\mathbf{u}}_c)^2](1 - e^2) \right\} \tag{15}$$

Using the relation $(\hat{\mathbf{e}} \cdot \hat{\mathbf{e}}_c)^2 + (\hat{\mathbf{e}} \cdot \hat{\mathbf{u}}_c)^2 + (\hat{\mathbf{e}} \cdot \hat{\mathbf{n}}_c)^2 = 1$ and a similar relation for $\hat{\mathbf{u}}$, this simplifies further to

$$H_{\text{sec}} = \frac{\mathfrak{G}\, m_c a^2}{4a_c^3(1 - e_c^2)^{3/2}} \left[-1 - \tfrac{3}{2}e^2 + \tfrac{3}{2}(\hat{\mathbf{e}} \cdot \hat{\mathbf{n}}_c)^2(1 + 4e^2) + \tfrac{3}{2}(\hat{\mathbf{u}} \cdot \hat{\mathbf{n}}_c)^2(1 - e^2) \right]. \tag{16}$$

We choose the reference frame to be the orbital plane of the companion star. Then $\hat{\mathbf{e}} \cdot \hat{\mathbf{n}}_c = \sin I \sin \omega$ and $\hat{\mathbf{u}} \cdot \hat{\mathbf{n}}_c = \sin I \cos \omega$ and

$$H_{\text{sec}} = \frac{\mathfrak{G}\, m_c a^2}{8a_c^3(1 - e_c^2)^{3/2}} \left[1 - 6e^2 - 3(1 - e^2)\cos^2 I + 15e^2\sin^2 I \sin^2 \omega \right]$$

$$= \frac{m_c L^4}{8\,\mathfrak{G}\, M^2 a_c^3(1 - e_c^2)^{3/2}} \left[\frac{6G^2}{L^2} - 5 - \frac{3H^2}{L^2} + 15\left(1 - \frac{G^2}{L^2}\right)\left(1 - \frac{H^2}{G^2}\right)\sin^2 \omega \right]. \tag{17}$$

The secular Hamiltonian is independent of ℓ so the semi-major axis is constant, and independent of Ω so H is constant. In contrast to the Hamiltonian (9), however, this Hamiltonian depends on the argument of periapsis ω, so the conjugate variable G (the angular momentum)

is *not* constant. This gives rise to the remarkable phenomenon of Lidov-Kozai oscillations, which we now describe.

Suppose that the initial planetary orbit is circular and inclined by I_0 to the reference plane (the orbital plane of the binary companion). Then $H = (GMa)^{1/2} \cos I_0$ and H and L are conserved, so conservation of H_{sec} requires

$$2 - 2e^2 + 5\sin^2\omega(e^2 - \sin^2 I_0) = 0 \quad \text{or} \quad e = 0. \tag{18}$$

It is not hard to show that the first of these equations has a solution for $0 < e < 1$ if and only if $I_0 > \sin^{-1}\sqrt{2/5} = 39.2°$. For larger initial inclinations the circular orbit may (and does) undergo oscillations to high eccentricity (**Lidov–Kozai oscillations**). The maximum eccentricity occurs at $\sin\omega = 1$ and is equal to

$$e_{\max} = \sqrt{\frac{5\sin^2 I_0 - 2}{3}}; \tag{19}$$

as I_0 approaches $90°$ the maximum eccentricity approaches unity (i.e., a collision orbit). Contours of equation (18 for $I_0 = 60°$ are shown in Figure 1.

Among the remarkable properties of Lidov–Kozai oscillations are that (i) above the critical inclination of $39.2°$, circular orbits are chaotic (because of additional terms due to octupole and higher moments of the tidal potential); (ii) the mass and distance of the companion star affect the period of the oscillations but not their amplitude. This means that in principle arbitrarily small external forces can lead to large changes in the orbit. In practice, when the perturber is sufficiently weak, either the oscillation period becomes longer than the age of the system, or other sources of precession—additional planets or general relativity—become stronger than the precession due to the companion, and this quenches the oscillations.

Note that there is an accidental degeneracy in the problem as formulated here: the Hamiltonian does not depend on the orientation of the apsis of the companion star. In more realistic treatments that include higher-order multipole moments this degeneracy is lifted, giving the dynamics an additional degree of freedom and thus more complicated behavior.

Lidov–Kozai oscillations are important in a remarkable variety of astrophysical contexts: the irregular satellites of the giant planets (Nesvorný et al. 2003, AJ 126, 398); the excitation of exoplanet eccentricities by companion stars (Holman et al. 1997, Nature 386, 254); the formation of close binary stars (Fabrycky & Tremaine 2007, ApJ 669, 1298); blue straggler stars (Perets & Fabrycky 2009, ApJ 697, 1048); Type Ia supernovae (Thompson 2011, ApJ 741, 82; Kushnir et al. 2013, ApJ 778, L37); planetary migration; black-hole mergers (Blaes et al. 2002, ApJ 578, 775); asteroid orbits; satellite orbits; comets; etc.
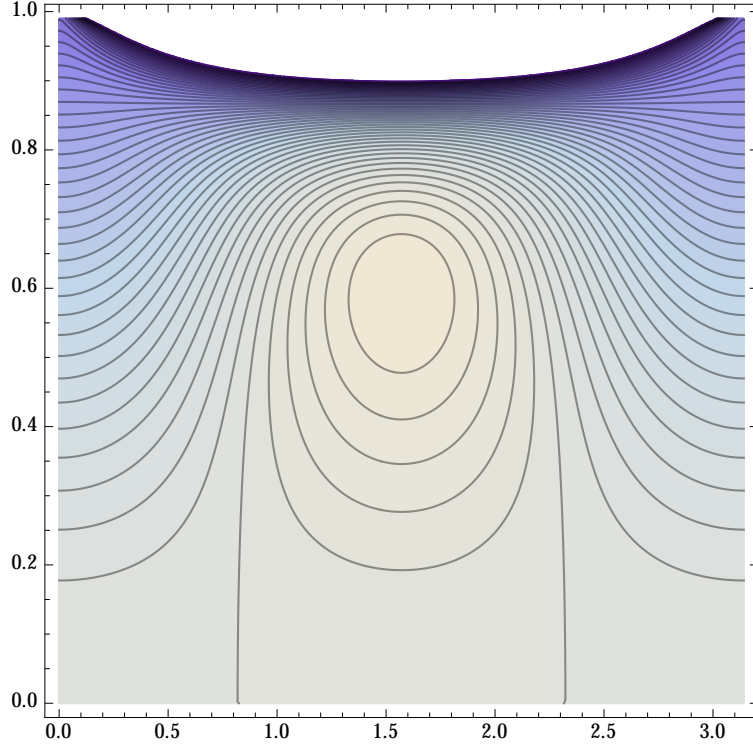
Fig. 1.— Lidov–Kozai oscillations of a planet that is initially in a circular orbit with inclination $I_0 = 60°$ to the orbit of a distant companion star. The vertical axis is eccentricity and the horizontal axis is argument of periapsis. The contours are level surfaces of the energy (18).

2.4. Mean-motion resonances

Orbital resonances are ubiquitous in the solar system (Peale 1976, ARAA 14, 215). Some examples include the 3:2 resonance between the orbital periods of Neptune and Pluto and other Kuiper belt objects; the 2:1 resonance between the orbital periods of Saturn's satellites Mimas and Tethys, and Dione and Enceladus; the co-orbital (1:1) resonance between Saturn's satellites Janus and Epimetheus; the Kirkwood gaps in the asteroid belt and the gaps in Saturn's rings; the three-body resonance between the mean motions of the first three Galilean satellites $(n_{\mathrm{Io}} - 3n_{\mathrm{Europa}} + 2n_{\mathrm{Ganymede}} = 0)$; Trojan asteroids; etc.

To keep the discussion as simple as possible, we focus on resonances between a planet on a circular orbit and a test particle orbiting on a near-circular orbit interior to the planet. We shall also assume that the orbits are coplanar. We denote the semi-major axis and mean motion of the planet's orbit by a_c and n_c, and its mass by m_c. According to the discussion

in §2.1 the disturbing function can be written

$$H_1 = \sum_{k,m,n} H_{kmn}(L, G, H) \cos(k\ell + m\omega + n\Omega - mn_st); \tag{20}$$

we have chosen the origin of time so that the planet is at azimuthal angle $\phi_c = 0$ at $t = 0$.

The Hamiltonian is further restricted by the d'Alembert property (5). We are assuming that the satellite and test-particle orbits are coplanar so $G = H$, which implies that the only non-zero terms have $n = m$. We are also assuming that the eccentricity of the test particle is small, so $H_{kmn} \sim e^{|m-k|}$. Thus the largest terms in the Hamiltonian have $k = m$, so we may approximate

$$H_1 = \sum_m A_m \cos[m(\ell + \omega + \Omega - n_ct)] = \sum_m H_m(L) \cos[m(\lambda - n_ct)]. \tag{21}$$

Resonance occurs when the rate of change of the argument of the cosine is small, that is, when $n \simeq n_c$. In other words the mean motions of the test particle and the planet must be nearly the same; examples of this resonance (also called a 1:1 resonance) include horseshoe orbits and Trojan orbits. These are quite interesting but we will not discuss them further because of time limitations.

The next strongest resonances occur when $H_{kmn} \sim e$ and require $m = k \pm 1$. For these H_1 can be approximated as

$$H_1 = \sum_m \sqrt{L - G} A_m \cos[(m \mp 1)\ell + m\omega + m\Omega - mn_ct]$$
$$= \sum_m \sqrt{L - G} A_m \cos[(m \mp 1)\lambda \pm \varpi - mn_ct] \tag{22}$$

where A_m is a constant (in principle A_m depends on L but since the variations in L are small we can treat A_m as a constant). Resonance occurs when $(m \mp 1)n = mn_c$. There are also resonances with $(m \mp j)n_c = mn_c$ with $j > 1$, but these are weaker since the resonant Hamiltonian is $\sim e^j$ by the d'Alembert property.

Since we have assumed that the test particle is interior to the planet, and since we can restrict ourselves without loss of generality to $m \geq 0$, we must choose the upper sign, so $n = mn_c/(m - 1)$ or $n/n_c = 2, \frac{3}{2}, \frac{4}{3}, \cdots$. These are known as **mean-motion** or **Lindblad** resonances.

We can simplify the problem by a canonical transformation using the generating function

$$S = -J_s[(m - 1)\ell + m\omega + m\Omega - mn_ct] + J_{f1}(\ell + \omega) + J_{f2}(\ell + \omega + \Omega). \tag{23}$$

The new coordinates are

$$\theta_s = \frac{\partial S}{\partial J_s} = -(m-1)\ell - m\omega - m\Omega + mn_c t, \quad \theta_{f1} = \ell + \omega, \quad \theta_{f2} = \ell + \omega + \Omega. \qquad (24)$$

The subscript "s" indicates that θ_s varies slowly near the resonance while "f" indicates that θ_{f1} and θ_{f2} vary rapidly ($\dot{\theta}_{f1} \simeq \dot{\theta}_{f2} \simeq n$). The new momenta are related to the Delaunay momenta by

$$L = \frac{\partial S}{\partial \ell} = (1-m)J_s + J_{f1} + J_{f2}, \quad G = \frac{\partial S}{\partial \omega} = -mJ_s + J_{f1} + J_{f2}, \quad H = -mJ_s + J_{f2}. \quad (25)$$

Solving for the new momenta,

$$J_s = L - G, \quad J_{f1} = G - H, \quad J_{f2} = H + m(L - G), \qquad (26)$$

and the new Hamiltonian is

$$H = H_K + H_1 + \frac{\partial S}{\partial t} = -\frac{(GM)^2}{2[(1-m)J_s + J_{f1} + J_{f2}]^2} + A_m\sqrt{J_s}\cos\theta_s + mn_c J_s. \qquad (27)$$

The Hamiltonian is independent of the fast coordinates θ_{f1} and θ_{f2} so J_{f1} and J_{f2} are constants. Since the inclination is zero $G = H$; thus $J_{f1} = 0$. Moreover since the eccentricity is small J_s is small and $J_{f2} \simeq L$, which is not small. Thus we can expand the denominator in equation (27) to quadratic order in J_s/J_{f2}. Dropping an unimportant constant term, we have

$$H = J_s[mn_c - (m-1)n_r] - \frac{3(m-1)^2}{2a_r^2}J_s^2 + A_m\sqrt{J_s}\cos\theta_s; \qquad (28)$$

here we have defined the constants a_r and n_r by $J_{f2} = (GMa_r)^{1/2}$ and $n_r = (GM/a_r^3)^{1/2}$. When the perturbing potential A_m is small and the eccentricity is also small (so J_s is small), we have $\dot{\theta}_s \simeq \partial H/\partial J_s = mn_c - (m-1)n_r$ so this constant can be thought of as the distance away from exact resonance, where $\dot{\theta}_s = 0$.

Rescaling the action and the Hamiltonian by constant factors allows us to convert the Hamiltonian into a standard form (Henrard & Lemaître 1983, Cel. Mech. 30, 197; Borderies & Goldreich 1984, Cel. Mech. 32, 127; Petrovich et al. 2013, ApJ 770, 24)

$$H = -3\Delta R + R^2 - 2\sqrt{2R}\cos r, \qquad (29)$$

where R and r is a conjugate momentum-coordinate pair and $\Delta \propto mn_c - (m-1)n_r$ measures the distance away from resonance. Henrard & Lemaître call this the "second fundamental model for resonance" in contrast to the "first fundamental model for resonance" which is the simple pendulum,

$$H = -\Delta R + \tfrac{1}{2}R^2 - \cos r. \qquad (30)$$

The Hamiltonian (29) has only one degree of freedom and thus is integrable. Thorough discussions of its solutions and how they evolve under slow changes in Δ (as might be caused, e.g., by migration or growth of the mass of the planet) are given in the references above. They provide an analytical description of such phenomena as resonance capture, eccentricity excitation during resonance crossing, etc. I shall not discuss these results due to lack of time, and since they are covered in the references. The main purpose of this discussion has been to show how perturbation theory can be used to derive a model Hamiltonian that is simple enough to solve analytically but realistic enough to capture the main features of mean-motion resonances.