# Generalized Global Symmetries

Nathan Seiberg
Institute for Advanced Study

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# Ordinary global symmetries

- Generated by operators associated with co-dimension one manifolds $M$
$$U_g(M)$$
$g \in G$ a group element
- The correlation functions of $U_g(M)$ are topological!
- Group multiplication $U_{g_1}(M)U_{g_2}(M) = U_{g_1 g_2}(M)$
- Local operators $O(p)$ are in representations of $G$
$$U_g(M)O_i(p) = R_i^j(g)O_j(p)$$
where $M$ surrounds $p$ (Ward identity)
- If the symmetry is continuous,
$$U_g(M) = e^{i \int j(g)}$$

$j(g)$ is a closed form current (its dual is a conserved current).

# $q$-form global symmetries

- Generated by operators associated with co-dimension $q+1$ manifolds $M$ (ordinary global symmetry has $q=0$)
$$U_g(M)$$
$g \in G$ a group element
- The correlation functions of $U_g(M)$ are topological!
- Group multiplication $U_{g_1}(M)U_{g_2}(M) = U_{g_1 g_2}(M)$. Because of the high co-dimension the order does not matter and $G$ is Abelian.
- The charged operators $\mathrm{V}(L)$ are on dimension $q$ manifolds $L$. Representations of $G$ – Ward identity
$$U_g(M)V(L) = R(g)V(L)$$
where $M$ surrounds $L$ and $R(g)$ is a phase.

# $q$-form global symmetries

If the symmetry is continuous,

$$U_g(M) = e^{i \int j(g)}$$

$j(g)$ is a closed form current (its dual is a conserved current).

Compactifying on a circle, a $q$-form symmetry leads to a $q$-form symmetry and a $q-1$-form symmetry in the lower dimensional theory.

- For example, compactifying a one-form symmetry leads to an ordinary symmetry in the lower dimensional theory.

No need for Lagrangian

- Exists abstractly, also in theories without a Lagrangian
- Useful in dualities

# $q$-form global symmetries

- Charged operators are extended (lines, surfaces)
- Charged objects are extended – branes (strings, domain walls)
  - In SUSY BPS bound when the symmetry is continuous

As with ordinary symmetries:

- Selection rules on amplitudes
- Couple to a background classical gauge field (twisted boundary conditions)
- Gauging the symmetry by summing over twisted sectors – like orbifolds.
  - Discrete $\theta$-parameters like discrete torsion.
- The symmetry could be spontaneously broken.
- There can be anomalies and anomaly inflow on defects.

# Example 1: $4d$ $U(1)$ gauge theory

Two global $U(1)$ one-form symmetries:

- Electric symmetry
  - Closed form currents: $\frac{2}{g^2} * F$ (measures the electric flux)
  - Shifts the gauge field $A$ by a flat connection
- Magnetic symmetry
  - Closed form currents: $\frac{1}{2\pi} F$ (measures the magnetic flux)
  - Shifts the magnetic gauge field by a flat connection. Nonlocal action on $A$.

# Example 1: $4d\ U(1)$ gauge theory

The symmetries are generated by surface operators

$$U_{g_E=e^{i\alpha},\, g_M=e^{i\eta}}(M) = e^{\frac{i\,\eta}{2\pi}\int F + \frac{2\,i\,\alpha}{g^2}\int *F}$$

- These are Gukov-Witten surface operators (rescaled $\alpha, \eta$).
- They measure the electric and the magnetic flux through the surface $M$.

The charged objects are dyonic lines

$$W_n(L) H_m(L)$$

($W_n(L)$ are Wilson lines and $H_m(L)$ are ‘t Hooft lines)

with global symmetry charges $n$ and $m$ under the two global $U(1)$ one-form symmetries.

# Example 2: $4d$ $U(1)$ gauge theory with charge $N$ scalars

The electric one-form global $U(1)$ symmetry is explicitly broken to $\boldsymbol{Z}_N$.

- Shifting by a flat $\boldsymbol{Z}_N$ connection does not affect the scalars.
- The Gukov-Witten operator $U$ is topological for any $\eta$, but $\alpha$ should be $2\pi k/N$.
- The charged operators are still

$$W_n(L)H_m(L)$$

- The explicit breaking of the global one-form electric symmetry to $\boldsymbol{Z}_N$ reflects the fact that the charge $N$ matter fields can screen $n$ in $W_n(L)$ and only $n \bmod (N)$ is interesting.

# Example 3: $4d$ $SU(N)$ gauge theory

- Electric $\boldsymbol{Z}_N$ one-form symmetry
  - The Gukov-Witten operator is associated with a conjugacy class in $SU(N)$. When this class is in the center of $SU(N)$ the surface operator is topological.
  - It shifts the gauge field by a flat $\boldsymbol{Z}_N$ connection.
  - It acts on the Wilson lines according to their representation under the $\boldsymbol{Z}_N \in SU(N)$ center.
- No magnetic one-form symmetry.
  - In this theory there are no ‘t Hooft lines – they are not genuine line operators.
  - There are open surface operators, whose boundaries are ‘t Hooft lines.

# Example 4: $4d\ SU(N)$ gauge theory with matter in $\boldsymbol{N}$

The presence of the charged matter explicitly breaks the electric one-form $\boldsymbol{Z}_N$ symmetry.

Hence, there is no global one-form symmetry.

# Example 5: $4d\ SU(N)/\boldsymbol{Z}_N$ gauge theory

- Here we gauged the electric one-form $\boldsymbol{Z}_N$ symmetry and hence it is no longer a global symmetry.
  - Since the charged Wilson lines are not gauge invariant, they are not genuine line operators – they need a surface.
- This theory has a discrete $\theta$-parameter. It can be absorbed in extending the range of the ordinary $\theta$-parameter (for spin manifolds) to $[0,2\pi N)$.
- There is a magnetic $\boldsymbol{Z}_N$ one-form symmetry.
  - The charge measures the ‘t Hooft flux through the surface.
  - The charged objects are the ‘t Hooft line $H$ and its powers.

# Significance of these symmetries

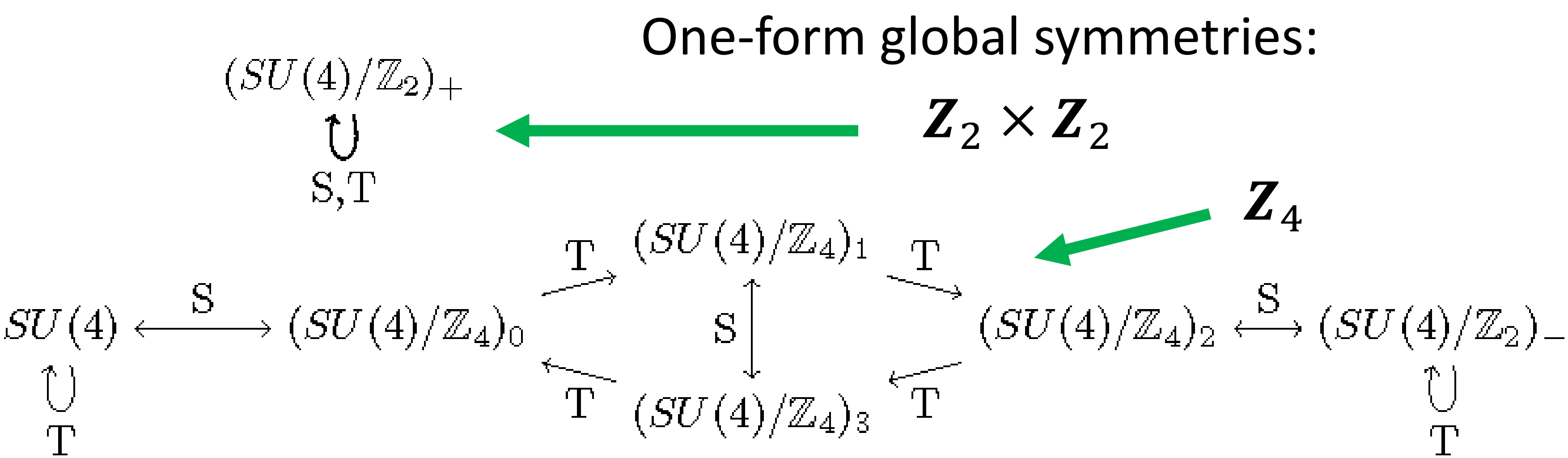
- Consequence: selection rules, e.g. in compact space the vev of a charged line wrapping a nontrivial cycle vanishes [Witten].
- Dual theories must have the same global symmetries. (They often have different gauge symmetries.)
  - The one-form symmetries are typically electric on one side of the duality and magnetic on the other.
  - $4d$ $\boldsymbol{N} = 1$ SUSY dualities respects the global symmetries.
  - The $SL(2, \boldsymbol{Z})$ orbit of a given $\boldsymbol{N} = 4$ theory must have the same global symmetry...

# Significance of these symmetries

– The $SL(2,\boldsymbol{Z})$ orbit of a given $\boldsymbol{N}=4$ theory must have the same global symmetry.

– Different $\boldsymbol{N}=4$ theories with the same gauge group (but different discrete $\theta$-parameter) can have different global symmetries. Then they must be on different $SL(2,\boldsymbol{Z})$ orbit.

– E.g.



# Significance of these symmetries

- Twisted sectors by coupling to flat background gauge fields
  - An $SU(N)$ gauge theory without matter can have twisted boundary conditions – an $SU(N)/\mathbf{Z}_N$ bundle, which is not an $SU(N)$ bundle – 't Hooft twisted boundary conditions.
- Gauging the symmetry by summing over twisted sectors – like orbifolds.
  - Discrete $\theta$-parameters are analogs of discrete torsion.
- Can characterize phases of gauge theories by whether the global symmetry is broken or not…

# Characterizing phases

- In a confining phase the electric one-form symmetry is unbroken.
  - The confining strings are charged and are classified by the unbroken symmetry.
- In a Higgs or Coulomb phase the electric one-form symmetry is broken.
  - Renormalizing the perimeter law to zero, the large size limit of $\langle W \rangle$ is nonzero – vev "breaks the symmetry."
  - It is unbroken in "Coulomb" phase in $3d$ and $2d$.

# Characterizing phases

- Generalizing known constraints about spontaneous symmetry breaking:
  - Continuous $q$-form global symmetries can be spontaneously broken only in more than $q+2$ dimensions.
  - Discrete $q$-form global symmetries can be spontaneously broken only in more than $q+1$ dimensions.

# Example 1: $4d$ $U(1)$ gauge theory

- There are two global $U(1)$ one-form symmetries.
- Both are spontaneously broken:
  - The photon is their Nambu-Goldstone boson
    $$\langle 0|F_{\mu\nu}|\epsilon, p\rangle = (\epsilon_\mu p_\nu - \epsilon_\nu p_\mu)e^{i\,px}$$
  - Placing the theory on $\boldsymbol{R}^3 \times \boldsymbol{S}^1$, each one-form global symmetry leads to an ordinary global symmetry and a one-form symmetry.
  - These ordinary symmetries are manifestly spontaneously broken – the moduli space of vacua is $\boldsymbol{T}^2$ parameterized by $A_4$ and the $3d$ dual photon.

# Example 2: $4d$ $U(1)$ gauge theory with charge $N$ scalars

When the scalars are massive the electric global $\boldsymbol{Z}_N$ and the magnetic one-form $U(1)$ symmetries are spontaneously broken.

- Accidental electric one-form $U(1)$ symmetry in the IR.

When the scalars condense and Higgs the gauge symmetry $U(1) \to \boldsymbol{Z}_N$ the spectrum is gapped.

- The electric $\boldsymbol{Z}_N$ global one-form symmetry is spontaneously broken.
  - It is realized in the IR as a $\boldsymbol{Z}_N$ gauge symmetry – long range topological order.
- The magnetic $U(1)$ global one-form symmetry is unbroken.
  - The strings are charged under it.

# Example 3: $4d$ $SU(N)$ gauge theory

In the standard confining phase the electric $\boldsymbol{Z}_N$ one-form symmetry is unbroken.

- Charged strings
- Area law in Wilson loops
- When compactified on a circle an ordinary ($q = 0$) $\boldsymbol{Z}_N$, which is unbroken [Polyakov]

If no confinement, the global $\boldsymbol{Z}_N$ symmetry is broken.

- No charged strings
- Perimeter law in Wilson loops
- When compactified on a circle an ordinary ($q = 0$) $\boldsymbol{Z}_N$, which is broken [Polyakov]

# Example 3: $4d\ SU(N)$ gauge theory

Can also have a phase with confinement index $t$, where the global one-form symmetry is spontaneously broken $\boldsymbol{Z}_N \to \boldsymbol{Z}_t$.

- $W$ has area law but $W^t$ has a perimeter law [Cachazo, NS, Witten].
- In this case there is a $\boldsymbol{Z}_{N/t}$ gauge theory at low energies – long range topological order.

# Example 4: $4d$ $SU(N)$ gauge theory with matter in $\boldsymbol{N}$

No global one-form symmetry.

Hence we cannot distinguish between Higgs and confinement.

This is usually described as screening the loop [Fradkin, Shenker; Banks, Rabinovici].

From our perspective, due to lack of symmetry.

# Example 5: $4d\ SU(N)/\boldsymbol{Z}_N$ gauge theory

Global magnetic $\boldsymbol{Z}_N$ one-form symmetry – the ‘t Hooft flux through the surface.

The order parameter is the ‘t Hooft loop $H$.

- In vacua with monopole condensation $H$ has a perimeter law. The magnetic $\boldsymbol{Z}_N$ is completely broken.
- In vacua with dyon condensation (oblique confinement) $H$ has an area law…

# Example 5: $4d\ SU(N)/\boldsymbol{Z}_N$ gauge theory

- $N$ different oblique confinement vacua labeled by the electric charge of the condensed dyon $p = 0, 1, \dots, N-1$.
- For nonzero $p$, $H$ has area law but $H^t$ (with $t = N/\mathrm{gcd}(p, N)$) has a perimeter law.
  - Correspondingly, the magnetic $\boldsymbol{Z}_N$ is broken $\boldsymbol{Z}_N \to \boldsymbol{Z}_t$.
  - The low energy theory has a $\boldsymbol{Z}_{\mathrm{gcd}(p,N)}$ gauge theory – long range topological order.
  - This is the magnetic version of a nontrivial confinement index $t$.
- $\boldsymbol{N} = 1$ SUSY $SU(N)/\boldsymbol{Z}_N$ gauge theory has $N$ vacua with $p = 0, 1, \dots, N-1$. They realize these phases.

# Example 6: $3d\ U(1)_N$

Global $\boldsymbol{Z}_N$ one-form symmetry

- Shift $A \to A + \frac{1}{N}\epsilon$ with $\epsilon$ a flat $U(1)$ gauge field with quantized periods.
- The Wilson lines are the charges.
- The Wilson lines are the charged objects.
- Cannot gauge this symmetry:
  – Gauging is like summing over insertions of the charge operators. But this makes everything zero.
  – The global $\boldsymbol{Z}_N$ one-form symmetry has 't Hooft anomaly.

# Higher Form SPT Phases

Consider a system with an unbroken symmetry with anomalies.

- ‘t Hooft anomaly matching forces excitations (perhaps only topological excitations) in the bulk, or only on the boundary.
- Symmetry Protected Topological Phase
- Domain walls between vacua in different SPT phases must have excitations.
- For examples, $\boldsymbol{N} = 1$ SUSY $SU(N)$ gauge theory has $N$ vacua in different SPT phases (the relevant symmetry is the one-form $\boldsymbol{Z}_N$ symmetry) and hence there is $U(k)_N$ on the domain walls between them [Acharya,Vafa]. Recent related work by [Dierigl, Pritzel].

# Conclusions

- Higher form global symmetries are ubiquitous.
- They help classify
  - extended objects (strings, domain walls, etc.)
  - extended operators/defects (lines, surfaces, etc.)
- As global symmetries, they must be the same in dual theories.
- They extend Landau characterization of phases based on order parameters that break global symmetries.
  - Rephrase the Wilson/'t Hooft classification in terms of broken or unbroken one-form global symmetries.
- Anomalies
  - 't Hooft matching conditions
  - Anomaly inflow
  - Degrees of freedom on domain walls

Thank you for your attention