Rigid SUSY in Curved Superspace

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Theme of recent developments: Rigid supersymmetric field theories in nontrivial spacetimes

- Relations between theories in different dimensions
- New computable observables in known theories
- New insights about the dynamics

Landscape of special cases

- $\mathcal{N} = 1$ on $\mathbf{AdS}_4$ [Zumino (77)…]
- $\mathcal{N} = 1, 2...$ on $\mathbf{S}^3 \times \mathbf{R}$ [D. Sen (87)…]
- $\mathcal{N} = 2$ on $\mathbf{S}^4$ [Pestun…]
- $\mathcal{N} = 1$ on $\mathbf{S}^3 \times \mathbf{S}^1$ [Romelsberger…]
- $\mathcal{N} = 2$ on $\mathbf{S}^3$ [Kapustin, Willett, Yaakov…]
- $\mathcal{N} = 2$ on $\mathbf{S}^2 \times \mathbf{S}^1$ [Kim; Imamura, Yokoyama…]

Questions/Outline

- How do we place a supersymmetric theory on a nontrivial spacetime?
  - When is it possible?
  - What is the Lagrangian?
  - How come we have supersymmetry on a sphere (or equivalently in *dS*)?
- How do we compute?
- What does it teach us?

SUSY in curved spacetime

- Naïve condition: need a covariantly constant spinor

$$\nabla_\mu \zeta = 0$$

- A more sophisticated condition: need a Killing spinor

$$\nabla_\mu \zeta = c\gamma_\mu \zeta$$

  with constant c .

- A more general possibility (also referred to as Killing spinor)

$$\nabla_\mu \zeta = \gamma_\mu \tilde{\zeta}$$

- Can include a background R gauge field in ∇_μ in any of these (twisting) [Witten…].

SUSY in curved spacetime

Motivated by supergravity: a more general condition

$$\nabla_\mu \zeta = M_\mu(x)\zeta$$

with an appropriate $M_\mu(x)$ (with spinor indices).

In the context of string or supergravity configurations $M_\mu(x)$ is determined by the background values of the various dynamical fields (forms, matter fields…).

All the dynamical fields have to satisfy their equations of motion.

Rigid SUSY in curved spacetime

$$\nabla_\mu \zeta = M_\mu(x) \zeta$$

We are interested in a rigid theory (no dynamical gravity) in curved spacetime:

- What is $M_\mu(x)$?
- Which constraints should it satisfy?
- Determine the curved spacetime supersymmetric Lagrangian.

Rigid SUSY in curved spacetime

We start with a flat space supersymmetric theory and want to determine the curved space theory.

- The Lagrangian can be deformed.
- The SUSY variation of the fields can be deformed.
- The SUSY algebra can be deformed.

Standard approach: Expand in large radius r and determine the correction terms iteratively in a power series in $1/r$.

- It is surprising when it works.
- In all examples the iterative procedure ends at order $1/r^2$.
- The procedure is tedious.

Landscape of special cases

- $\mathcal{N}=1$ on $\mathbf{AdS}_4$ [Zumino (77)…]
- $\mathcal{N}=1,2...$ on $\mathbf{S}^3\times\mathbf{R}$ [D. Sen (87)…]
- $\mathcal{N}=2$ on $\mathbf{S}^4$ [Pestun…]
- $\mathcal{N}=1$ on $\mathbf{S}^3\times\mathbf{S}^1$ [Romelsberger…]
- $\mathcal{N}=2$ on $\mathbf{S}^3$ [Kapustin, Willett, Yaakov…]
- $\mathcal{N}=2$ on $\mathbf{S}^2\times\mathbf{S}^1$ [Kim; Imamura, Yokoyama…]

All these backgrounds are conformally flat.

So it is straightforward to put an SCFT on them.

Example: the partition function on $\mathbf{S}^3\times\mathbf{S}^1$ is the superconformal index [Kinney, Maldacena, Minwalla, Raju].

But for non-conformal theories it is tedious and not conceptual.

What is the most general setup?

Main point

- Nontrivial background metric should be viewed as part of a background superfield.
- Study SUGRA in superspace and view the fields in the gravity multiplet as arbitrary, classical, background fields.
- Do not impose any equation of motion; i.e. $M_P \to \infty$
- Metric and auxiliary fields are on equal footing.
- Most of the terms in the supergravity Lagrangian including the graviton kinetic term are irrelevant.
- Then, supersymmetry is preserved provided the metric and the auxiliary fields satisfy certain conditions (below).

Linearized supergravity

Simplest $M_P \to \infty$ limit is linearized supergravity

$$\mathcal{L} = \mathcal{L}_{\text{flat space}} + h^{\mu\nu} T_{\mu\nu} + \psi^{\mu\alpha} S_{\mu\alpha} + \bar{\psi}^{\mu\dot{\alpha}} \bar{S}_{\mu\dot{\alpha}}$$

$$+ b^{\mu} j_{\mu} + \bar{M} X + M \bar{X} + \ldots$$

$T_{\mu\nu}, \; S_{\mu\alpha}, \; \bar{S}_{\mu\dot{\alpha}}, \; j_{\mu}, \; X, \; \bar{X}$ are operators in the SUSY multiplet of the energy momentum tensor. They are constructed out of the matter fields.

$h_{\mu\nu}, \; \psi_{\mu\alpha}, \; \bar{\psi}_{\mu\dot{\alpha}}$ are the deviation of the metric and the gravitino.

$b^{\mu}, \; M$ are a vector and a complex scalar auxiliary fields.

The Rigid Limit

- We are interested in spaces with arbitrary metric, so we need a more subtle $M_P \to \infty$ limit – the rigid limit.

- Take $M_P \to \infty$ with fixed metric and appropriate scaling (weight one) of the various auxiliary fields in the gravity multiplet:

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_2$$

$\mathcal{L}_0$ is obtained from the flat space theory by inserting the curved space metric – the naïve term.

$\mathcal{L}_1$ is linear in the auxiliary fields – as in linearized SUGRA.

$\mathcal{L}_2$ arises from the curvature and terms quadratic in the auxiliary fields – “seagull terms” for SUSY.

The Rigid Limit Lagrangian

For example, the bosonic terms in a Wess-Zumino model with K and W are

$$\frac{1}{e}\mathcal{L}_0 = K_{i\bar{j}}\left(F^i\bar{F}^{\bar{j}} - \partial_\mu\bar{\phi}^{\bar{j}}\partial^\mu\phi^i\right) + F^iW_i + \bar{F}^{\bar{j}}\bar{W}_{\bar{j}}$$

$$\frac{1}{e}\mathcal{L}_1 = -M\left(\frac{1}{3}K_iF^i + \bar{W}\right) - \bar{M}\left(\frac{1}{3}K_{\bar{i}}\bar{F}^{\bar{i}} + W\right) - \frac{i}{3}b^\mu\left(K_i\partial_\mu\phi^i - K_{\bar{i}}\partial_\mu\bar{\phi}^{\bar{i}}\right)$$

$$\frac{1}{e}\mathcal{L}_2 = \left(\frac{1}{6}\mathcal{R} + \frac{1}{9}M\bar{M} - \frac{1}{9}b_\mu b^\mu\right)K$$

Supersymmetric backgrounds

For supersymmetry, ensure that the variation of the gravitino vanishes

$$\begin{aligned} 2\nabla_\mu \zeta^\alpha &= \frac{i}{3}\left(M(\epsilon\sigma_\mu\bar{\zeta})^\alpha + 2b_\mu\zeta^\alpha + 2b^\nu(\zeta\sigma_{\nu\mu})^\alpha\right) \\ 2\nabla_\mu \bar{\zeta}_{\dot{\alpha}} &= -\frac{i}{3}\left(\bar{M}(\zeta\sigma_\mu)_{\dot{\alpha}} + 2b_\mu\bar{\zeta}_{\dot{\alpha}} + 2b^\nu(\bar{\zeta}\bar{\sigma}_{\nu\mu})_{\dot{\alpha}}\right) \end{aligned}$$

- These conditions depend only on the metric and the auxiliary fields in the gravity multiplet.
- They are independent of the dynamical matter fields.
- In Euclidean space bar does not mean c.c.

Curved superspace

- For supersymmetry $\nabla_\mu \zeta = M_\mu(x)\zeta$
- Integrability condition: differential equations for the metric and the various auxiliary fields through $M_\mu(x)$.
- The supergravity Lagrangian with nonzero background fields gives us a rigid field theory in curved superspace.
- Comments:
  - Enormous simplification
  - This makes it clear that the iterative procedure in powers of $1/r$ terminates at order $1/r^2$.
  - Different off-shell formulations of supergravity (which are equivalent on-shell) can lead to different backgrounds.

An alternative formalism

- If the rigid theory has a continuous global R-symmetry, there is an alternative supergravity formalism known as "new minimal supergravity" (the previous discussion used "old minimal supergravity").
- Here the auxiliary fields are a $U(1)_R$ gauge field A_μ and a two form $B_{\mu\nu}$.
- On-shell this supergravity is identical to the standard one. But since we do not impose the equations of motion, we should treat it separately.
- The familiar topological twisting of supersymmetric field theories amounts to a background A_μ in this formalism.
- The expressions for the rigid limit and the conditions for unbroken supersymmetry are similar to the expressions above.

Examples: $\mathbf{AdS}_4$ and $\mathbf{S}^4$

- $\mathbf{AdS}_4$: turn on a constant value of a scalar auxiliary field

$$M = \frac{3}{r}$$

- $\mathbf{S}^4$: set the auxiliary fields

$$M = \overline{M} = \frac{3}{ir}$$

  - Note: not the standard reality!
  - Equivalently, $r \to ir$ in Euclidean $\mathbf{AdS}_4$.
  - When non-conformal, not reflection positive (non-unitary). Hence, consistent with "no SUSY in *dS*."
  - In terms of the characteristic mass scale *m* and the radius *r* the problematic terms are of order *m/r.*

Examples: $\mathbf{AdS}_4$ and $\mathbf{S}^4$

- In these two examples the superalgebra is $OSp(1|4)$
  - Good real form for Lorentzian $\mathbf{AdS}_4$
  - As always in Euclidean space, $Q^* \neq \bar{Q}$.
  - For $\mathbf{S}^4$ need a compact real form of the isometry

$$SO(5) \sim Sp(4) \subset OSp(1|4)$$

  - Then, the anti-commutator of two supercharges is not a real rotation.
  - Hence, hard to compute using localization.
- The superpotential is not protected (can be absorbed in the Kahler potential) and holomorphy is not useful.
- For *N=2* the superalgebra is $OSp(2|4)$
  - computable

Example: *N=1* on $\mathbf{S}^3 \times \mathbf{S}^1$

- Turn on a vector auxiliary field along $\mathbf{S}^1$

$$b_\mu = \frac{i}{r}\delta_{\mu 4}$$

- For *Q* to be well defined around the $\mathbf{S}^1$, need a global continuous R-symmetry and a background $U(1)_R$ gauge field.
- Supersymmetry algebra: $SU(2|1) \times SU(2) \times U(1)$, where the $U(1)$ factor is the combination of “time” translation and R-symmetry that commutes with *Q*.
- Alternatively, can use “new-minimal” supergravity and turn on a $U(1)_R$ gauge field and a constant *H=dB* on $\mathbf{S}^3$, where *B* is a two-form auxiliary field.
- No quantization conditions on the periods of the auxiliary fields.

Deforming the theory

On $\mathbf{S}^3 \times \mathbf{R}$ (or $\mathbf{S}^3 \times \mathbf{S}^1$) we can add background gauge fields for the non-R flavor symmetries, $U(1)_f$; turn on constant complex A^f along $\mathbf{R}$:

- $\mathrm{Re}\, A^f$ leads to a real mass in the *3d* theory on $\mathbf{S}^3$.
- $\mathrm{Im}\, A^f$ shifts the choice of R-symmetry by $U(1)_f$.
- The partition function is manifestly holomorphic in A^f.

We can also squash the $\mathbf{S}^3$. We will not pursue it here.

The partition function on $\mathbf{S}^3 \times \mathbf{S}^1$

- It is a trace over a Hilbert space with (complex) chemical potentials A^f .
- Only short representations of $SU(2|1)$ contribute to the trace [Romelsberger].
  - Note, this is an index, but in general it is not "the superconformal index."
- It is independent of small changes in the parameters of the *4d* Lagrangian – it has the same value in the UV and IR theories.
- It is holomorphic in A^f.

$\mathcal{N}=1$ on $\mathbf{S}^3 \times \mathbf{S}^1$

- If the theory is conformal, the partition function is the superconformal index.
- For non-conformal theories the partition function does not depend on the scale [Romelsberger].
- Can use a free field computation in the UV to learn about the IR answer. (Equivalently, use localization.)
- This probes the operators in short representations and their quantum numbers (more than just the chiral ones).
- Highly nontrivial information about the IR theory; *e.g.* can test dual descriptions of it [Romelsberger, Dolan, Osborn, Spiridonov, Vartanov…].

Answers

A typical expression [Dolan, Osborn]

$$I_E(p,q,\mathrm{y},\tilde{\mathrm{y}})_{SU(N_c)}$$
$$= (p;p)^{N_c-1}(q;q)^{N_c-1}\frac{1}{N_c!}\int\prod_{j=1}^{N_c-1}\frac{\mathrm{d}z_j}{2\pi i z_j}\frac{\prod_{1\le i\le N_f}\prod_{1\le j\le N_c}\Gamma(y_i z_j, 1/(\tilde{y}_i z_j);p,q)}{\prod_{1\le i<j\le N_c}\Gamma(z_i/z_j, z_j/z_i;p,q)}\Bigg|_{\prod_{j=1}^{N_c} z_j=1}$$

Γ Elliptic gamma function

General lessons:

- Very explicit
- Nontrivial
- Special functions – relation to the elliptic hypergeometric series of [Frenkel, Turaev]
- To prove duality, need miraculous identities [Rains, Spiridonov...]

Example: N=2 with $U(1)_R$ on $\mathbf{S}^3$

- Can consider as a limit of the previous case

$$\mathbf{S}^3 \times \mathbf{S}^1 \to \mathbf{S}^3$$

- Can also view as a *3d* theory, where we can add new terms, *e.g.* Chern-Simons terms.
- Nonzero *H=dB* ensures supersymmetry.
- Supersymmetry algebra: $SU(2|1) \times SU(2)$
- As in the theory on $\mathbf{S}^4$, if the theory is not conformal, it is not unitary. (No SUSY in *dS* space.)
  – In terms of the characteristic mass scale *m* and the radius *r* the problematic terms are of order *m/r.*

Example: $\mathcal{N}=2\ U(1)$ on $\mathbf{S}^3$

[Kapustin, Willett, Yaakov…]

$$\frac{1}{e}\mathcal{L}=\frac{1}{g_{YM}^2}\Big(\frac{1}{2}F_{\mu\nu}^2+(\partial_\mu\sigma)^2-D^2+i\lambda^\dagger\nabla\!\!\!/_\mu\lambda$$

$$+\frac{2i}{r}D\sigma-\frac{1}{2r}\lambda^\dagger\lambda+\frac{1}{r^2}\sigma^2\Big)$$

- The $\mathcal{O}(1/r)$ terms are not reflection positive (non-unitary).
- Since the answer is independent of g_{YM}, we can take it to zero and find that the theory localizes on

$$D=\frac{i}{r}\sigma=\text{const.}\qquad F_{\mu\nu}=0$$

- The one loop determinant is computable.

Generalizations

- Non-Abelian theories
- Add matter fields
- Add Chern-Simons terms
- Add Wilson lines

In all these cases the functional integral becomes a matrix model for σ .

The partition function and some correlation functions of Wilson loops are computable.

Duality in *3d* *N*=2,…

In *3d* there are very few diagnostics of duality/mirror symmetry.

The partition functions on $\mathbf{S}^3$, $\mathbf{S}^2 \times \mathbf{S}^1$ provide new nontrivial tests.

Examples (similar to duality in *4d*):

- *3d* mirror symmetry [Intriligator, NS…] was tested [Kapustin, Willett, Yaakov]
- The $U(N_c)_k \ / \ U(|k| + N_f - N_c)_{-k}$ duality of [Aharony; Giveon and Kutasov] was tested [Kapustin, Willett, Yaakov; Bashkirov …].
- Generalizations [Kapustin]:

$$USp(2N_c)_k \ / \ USp(2(|k| + N_f - N_c - 1))_{-k}$$

$$O(N_c)_k \ / \ O(|k| + N_f - N_c + 2)_{-k}$$

Z-minimization

- Consider an *N=2 3d* theory with an R-symmetry and some non-R-symmetries $U(1)_f$ with charges F_f .
- If there are no accidental symmetries in the IR theory, the R-symmetry in the superconformal algebra at the IR fixed point is a linear combination of the charges

$$R_{IR} = R + \sum a^f F_f$$

- In *4d* the coefficients a^f are determined by *a*-maximization [Intriligator, Wecht].
- What happens in *3d*?

Z-minimization

$$R_{IR} = R + \sum a^f F_f$$

- The partition function $Z(\mathbf{S}^3)$ can be studied as a function of a^f [Jafferis; Hama, Hosomichi, Lee].
  (Recall, $a^f = \mathrm{Im}\, A^f$ can be introduced as a complex background $U(1)_f$ gauge field.)
- Jafferis conjectured that $|Z(\mathbf{S}^3)|$ is minimized at the IR values of a^f.
- Many tests
- Extension of *4d a*-maximization.
- Is there a version of a *c*-theorem in *3d*?

Conclusions

- The rigid limit of supergravity leads to field theories in curved superspace.
- When certain conditions are satisfied the background is supersymmetric. Then, a simple, unified and systematic procedure leads to
  - The supersymmetric Lagrangian
  - The superalgebra
  - The variations of the fields
- A rich landscape of rigid supersymmetric field theories in curved spacetime was uncovered.
- Many observables were computed leading to new insights about the dynamics.