Anomalies, Conformal Manifolds, and Spheres

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Jaume Gomis, Po-Shen Hsin, Zohar Komargodski, Adam Schwimmer, NS, Stefan Theisen, arXiv:1509.08511

CFT Sphere partition function $\log Z$

- Power divergent terms are not universal. Can be removed by counter terms like $\Lambda^d \int \sqrt{g} + \Lambda^{d-2} \int \sqrt{g} R +$ ...
- In addition
  - for odd *d* $\log Z = -F$ is universal (ambiguity in quantized imaginary part due to Chern-Simons terms – depends on framing)
  - for even *d* $\log Z = C \log(r\Lambda) - F$.
    - C is universal (in *2d* it is $c/3$ and in *4d* it is – a)
    - F is not universal; it can be absorbed in a local counter-term, $\int \sqrt{g}\, E_d\, F$ with E_d the Euler density
- Used in c-theorem and its generalizations, entanglement entropy, ...

Conformal manifolds

- S$= S_0 + \int \lambda^i O_i(x)$ is an exactly marginal deformation
- Family of CFTs labeled by coordinates λ^i
- Metric on the conformal manifold – the Zamolodchikov metric

$$\langle\, O_i(x)\, O_j(0)\,\rangle = \frac{g_{ij}(\lambda)}{|x|^{2d}}$$

- Focus on $d = 2$

Conformal manifolds of *2d* (2,2) SCFT

- Typical example: sigma models with Calabi-Yau target space.
- In string theory the coordinates on the conformal manifold are the moduli – massless fields in *4d*
  - *2d* chiral fields λ
  - *2d* twisted chiral fields $\tilde{\lambda}$
- The conformal manifold is Kahler with

$$K = K_c(\lambda, \bar{\lambda}) + K_{tc}\left(\tilde{\lambda}, \bar{\tilde{\lambda}}\right)$$

2d (2,2) curved superspace

- For rigid SUSY in curved spacetime use supergravity [Festuccia, NS]
- Simplification in *2d*
  - (locally) pick conformal gauge $g_{\mu\nu} = e^{2\sigma}\delta_{\mu\nu}$.
  - for SUSY (locally) pick superconformal gauge.
- Two kinds of (2,2) supergravities. In the superconformal gauge they depend on
  - A chiral $\Sigma = \sigma + i\,a + \cdots$ with $A_\mu = \epsilon_{\mu\nu}\partial^\nu a$ an axial R-gauge field (Lorentz gauge). We will focus on this.
  - A twisted chiral $\tilde{\Sigma} = \sigma + i\,\tilde{a} + \cdots$ with $\widetilde{A_\mu} = \epsilon_{\mu\nu}\partial^\nu \tilde{a}$ a vector R-gauge field (Lorentz gauge).

2d (2,2) curved superspace

- In this language the curvature is a chiral superfield $\mathcal{R} = \bar{D}^2\bar{\Sigma}$
- To preserve rigid SUSY in a non-conformal theory we need to add terms to the flat superspace Lagrangian
- Various backgrounds in the literature are easily described, e.g.
  - Topological twist is $\Sigma = 0,\ \bar{\Sigma} \neq 0$
  - Omega background
  - Supersymmetry on $\mathbb{S}^2$ [Benini, Cremonesi; Doroud, Gomis, Le Floch, Lee] is achieved with (suppress r dependence)

$$\Sigma = -\log(1 + |z|^2) + \theta^2 \frac{i}{1+|z|^2}$$

$$\bar{\Sigma} = -\log(1 + |z|^2) + \bar{\theta}^2 \frac{i}{1+|z|^2}$$

  - $\bar{\Sigma}$ is not necessarily the complex conjugate of Σ

(2,2) sphere partition functions

- Amazing conjecture [Jockers, Kumar, Lapan, Morrison, Romo]: the $\mathbb{S}^2$ partition function with the background of [Benini, Cremonesi; Doroud, Gomis, Le Floch, Lee] is

$$Z = \ r^{c/3} e^{-K_c(\lambda,\bar{\lambda})}$$

(restoring the radius r, whose power reflects the ordinary conformal anomaly).

Similarly, using $\tilde{\Sigma}$ it is $\mathrm{Z} = r^{c/3} e^{-K_{tc}(\tilde{\lambda},\bar{\tilde{\lambda}})}$.

- Proofs based on localization, squashed sphere, *tt**, twisting, counter terms and properties of the background [Gomis, Lee; Gerchkovitz, Gomis, Komargodski; ...]

Questions/confusions

- Given that the one point function of a marginal operator vanishes, how can the sphere partition function depend on λ?
- Why is it meaningful?
  - Can add a local counter term $\int \sqrt{g} R\, f\left(\lambda, \bar{\lambda}\right)$, making the answer non-universal
- In an SCFT on the sphere there is no need to add terms to the Lagrangian to preserve SUSY
  - Why does it depend on the background Σ?
  - If it does not, what determines whether we used Σ or $\tilde{\Sigma}$ to find e^{-K_c} or $e^{-K_{tc}}$?
  - Where is the freedom in Kahler transformations?
- What's the conceptual reason for it? Is it UV or IR?

Conformal manifolds and anomalies (w/o SUSY)

Zamolodchikov metric $\langle O_i(x)\, O_j(0)\rangle = \frac{g_{ij}(\lambda)}{|x|^4}$

In momentum space $\int e^{ipx}\langle O_i(x)\, O_j(0)\rangle \sim g_{ij}(\lambda)p^2 \log(\mu^2/p^2)$

Dependence on the scale μ leads to a conformal anomaly: with position dependent λ the variation with respect to the conformal factor $\delta\sigma$ is (suppressed coefficients) [Osborn; Friedan, Konechny]

$$\delta S_{eff} \sim \int \delta\sigma \left(c\, R + g_{ij}(\lambda)\, \partial_\mu \lambda^i\, \partial^\mu \lambda^j + \ \ldots\right)$$

Ordinary conformal anomaly A more subtle anomaly

(2,2) SCFT

- The anomaly should be expressed in superspace – need to use curved superspace.
- Focus on the supergravity that in the superconformal gauge uses a chiral superfield Σ
- The supersymmetrization of the anomaly is

$$\delta S_{eff} \sim \int d^2x\, d^4\theta\, (\delta\Sigma + \delta\bar{\Sigma})\left(c\,(\Sigma + \bar{\Sigma}) + K_c(\lambda, \bar{\lambda}) - K_{tc}\left(\tilde{\lambda}, \bar{\tilde{\lambda}}\right)\right)$$

- It is invariant under Kahler transformations of $K_{tc}\left(\tilde{\lambda}, \bar{\tilde{\lambda}}\right)$, but not under Kahler transformations of $K_c(\lambda, \bar{\lambda})$.

Ambiguities

- In the superconformal gauge the local terms are expressed in terms of the chiral curvature superfield $\mathcal{R} = \bar{D}^2\bar{\Sigma}$.
- Terms that depend on Σ not through $\mathcal{R}$ are non-local.
- Therefore, the anomaly is not a variation of a local term.
- Freedom in the local term

$$\int d^2\theta\, \mathcal{R}\, f(\lambda) + c.c. = \int d^4\theta\, \bar{\Sigma} f(\lambda) + c.c.$$

leads to freedom in Kahler transformations of $K_c(\lambda\,,\bar{\lambda})$. Other than, that the anomaly is unambiguous.

The anomaly in components

For a purely conformal variation $\delta\Sigma = \delta\sigma$ the anomaly is

$$\delta S_{eff} \sim \int d^4\theta \, (\delta\Sigma + \delta\bar{\Sigma}) \left(c \, (\Sigma + \bar{\Sigma}) + K_c(\lambda\,, \bar{\lambda}) - K_{tc}\left(\tilde{\lambda}, \bar{\tilde{\lambda}}\right)\right)$$

$$= \int \Big[\delta\sigma \, \left(c \, \Box\sigma + g_{i\bar{\imath}} \partial_\mu \lambda^i \partial^\mu \bar{\lambda}^{\bar{\imath}} + \tilde{g}_{a\bar{a}} \, \partial_\mu \tilde{\lambda}^a \partial^\mu \bar{\tilde{\lambda}}^{\bar{a}} \right)$$

$$- \Box\delta\sigma \, K_c(\lambda\,, \bar{\lambda}) \Big]$$

The last term leads to

$$S_{eff} \sim \int R \, K_c(\lambda, \bar{\lambda}) + \ \ldots$$

and hence on $\mathbb{S}^2$

$$Z = \ r^{c/3} e^{-K_c}$$

Q.E.D.

Extensions

- Trivial to repeat with $\tilde{\Sigma}$ and to find K_{tc}
- Can reproduce that for *4d* $\mathcal{N} = 2$ [Gerchkovitz, Gomis, Komargodski; Gomis Ishtiaque]

$$Z = r^{-a}\, e^{K(\tau,\bar{\tau})/12}$$

Ordinary conformal anomaly

The more subtle anomaly

- *2d* $\mathcal{N} = (0,2)$

Conclusions

- Anomaly under conformal transformations when the coupling constants depend on position
  - Unrelated to supersymmetry
  - This is a UV phenomenon
    - Visible on flat $\mathbb{R}^2$
    - Independent of the background
- Supersymmetry restricts
  - the form of the anomaly
  - the ambiguity due to local counter terms

Conclusions

- The $\mathbb{S}^2$ partition function depends the anomaly.
  - New derivation of

$$Z = r^{c/3} e^{-K_c}$$

    - The usual conformal anomaly
    - The more subtle conformal anomaly
  - Addresses the questions/confusions we raised
- Three step process
  - the anomaly
  - the ambiguity (freedom in counter terms)
  - the sphere
- Several extensions