Adventures with Contact Terms

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Outline

Throughout the talk we will be in $3d$.

- ► Chern-Simons contact terms
- ► Currents in $\mathcal{N} = 2$ supersymmetry
- ► Various SUSY Chern-Simons contact terms
- ► Anomaly: superconformal vs. (compact) $U(1)_R$ symmetry
- ► The partition function of $\mathcal{N} = 2$ on a three sphere
- ► F-maximization
- ► Tests of duality
- ► Conclusions

Contact Terms

Contact terms are correlation functions at coincident points.

- ▶ Some of them are determined; e.g.
  - ▶ The seagull term (needed for gauge invariance)
  - ▶ The $2d$ conformal anomaly (in a CFT $T_\mu^\mu = 0$, but it must have nonzero contact terms)
- ▶ Most of them are arbitrary (not universal).
  - ▶ They reflect short distance physics.
  - ▶ They depend on the regularization scheme.
  - ▶ They are associated with local counterterms constructed out of the dynamical fields and background fields.
  - ▶ They change under dynamical and background fields redefinitions (coupling constants redefinitions).

An Important Exception

Consider a three-dimensional field theory with a global (compact) $U(1)$ symmetry.

The conserved current j_μ can be coupled to a classical background $U(1)$ gauge field a_μ.

A contact term in the two-point function

$$\langle j_\mu(x) j_\nu(0) \rangle = \cdots + \frac{i\kappa}{2\pi} \epsilon_{\mu\nu\rho} \partial^\rho \delta^{(3)}(x)$$

can be interpreted as due to a Chern-Simons (CS) counterterm in the background fields

$$\frac{i\kappa}{4\pi} \int \varepsilon^{\mu\nu\rho} a_\mu \partial_\nu a_\rho \ .$$

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- However, since $U(1)$ is compact, and we would like the theory to make sense on arbitrary manifolds with arbitrary $U(1)$ bundles, the freedom in κ is quantized.

- Sometime, a consistent definition of the theory forces us to add such a bare counterterm with fixed value of $\kappa \, \mathrm{mod}(1)$; still, the integer part of κ is arbitrary.

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- Sometime, a consistent definition of the theory forces us to add such a bare counterterm with fixed value of $\kappa \bmod(1)$; still, the integer part of κ is arbitrary.
- $\kappa \bmod(1)$ is a physical observable.

Technical comments:

- ▸ We study a theory on a spin manifold (otherwise, the quantization of κ is different).
- ▸ We need to define the Chern-Simons term by extending the fields to $4d$. Then the quantization of κ follows from imposing independence of the extension.

We will have these comments in mind, but will not repeat them.

The Current Two-Point Function

Current conservation restricts

$$\int e^{ip\cdot x} \langle j_\mu(x) j_\nu(0) \rangle = \tau\left(\frac{p^2}{\mu^2}\right) \frac{p_\mu p_\nu - p^2 \delta_{\mu\nu}}{16|p|} + \kappa\left(\frac{p^2}{\mu^2}\right) \frac{\varepsilon_{\mu\nu\rho} p^\rho}{2\pi}$$

- Two structure functions
- In a conformal field theory τ and κ are independent of p.
- In a unitary CFT $\tau > 0$.
- The p dependence of $\kappa\left(\frac{p^2}{\mu^2}\right)$ is physical – not a contact term.
- Shifting $\kappa\left(\frac{p^2}{\mu^2}\right)$ by a constant amounts to adding a contact term.
- If the symmetry is compact, this ambiguity is quantized.

We define

$$\kappa_{UV} \equiv \lim_{p \to \infty} \kappa \left(\frac{p^2}{\mu^2} \right)$$

$$\kappa_{IR} \equiv \lim_{p \to 0} \kappa \left(\frac{p^2}{\mu^2} \right)$$

They can be changed by adding a local counterterm, but $\kappa_{UV} - \kappa_{IR}$ is physical.

The same story can be repeated for the energy-momentum tensor and the Lorentz Chern-Simons term

$$\frac{i}{4} \int \varepsilon^{\mu\nu\rho} \operatorname{Tr} \left(\omega_\mu \partial_\nu \omega_\rho + \frac{2}{3} \omega_\mu \omega_\nu \omega_\rho \right) \quad ,$$

or more precisely, its definition in terms of a $4d$ extension.

Example 1: Free Fermions

A theory with a single massive fermion has a global $U(1)$ symmetry with [Redlich]

$$\kappa_{UV} - \kappa_{IR} = \frac{1}{2}\text{sign}(m) \ .$$

The IR theory has no degrees of freedom.
The effective Lagrangian is proportional to a Chern-Simons term for the background $U(1)$ gauge field.
More precisely, the functional integral leads to the η-invariant, which is discontinuous.

Consistency demands $\kappa_{IR} \in \mathbf{Z}$, and therefore we shift the UV theory by a counterterm such that

$$\kappa_{UV} = \frac{1}{2} + \text{integer} \ .$$

Example 2: A Topological theory

Consider a theory of a dynamical $U(1)$ gauge field A_μ and a classical $U(1)$ gauge field a_μ

$$\mathcal{L} = \frac{i}{4\pi} \left(k\, \varepsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho + 2p\, \varepsilon^{\mu\nu\rho} a_\mu \partial_\nu A_\rho + q\, \varepsilon^{\mu\nu\rho} a_\mu \partial_\nu a_\rho \right)$$

with $k, p, q \in \mathbf{Z}$.

Naively integrating out A_μ, we find the effective theory

$$\mathcal{L}_{eff} = \frac{i\kappa}{4\pi} \varepsilon^{\mu\nu\rho} a_\mu \partial_\nu a_\rho$$

$$\kappa = \kappa_{UV} = \kappa_{IR} = q - \frac{p^2}{k}$$

- κ is fractional, but the theory is consistent.
- There are remaining topological IR degrees of freedom. Hence, A_μ was not integrated out properly.

Example 3: Flowing From a Free Theory to a Massive Theory

Computing in perturbation theory and taking into account only 1PI graphs:

- The subtlety in the previous example does not affect κ^{1PI}.
- κ_{UV}^{1PI} must be integer or half-integer.
- Non-renormalization theorem [Coleman, Hill]:
  κ_{IR}^{1PI} can be generated only at one loop.
  Alternatively, κ_{IR}^{1PI} must be quantized and then the theorem follows from gauge invariance.

Example 4: Interesting Renormalization Group Flow

We often have a free theory in the UV.
The IR theory is fully gapped (not even topological d.o.f).
At energies E such that $m \ll E \ll M$ the theory is approximately conformal. M and m are crossover scales.

- $\kappa_{UV} = \lim_{p \to \infty} \kappa(p^2)$ is determined modulo an integer by the number of fermions and the coupling to topological terms.

- $\kappa_{CFT} = \kappa(m^2 \ll p^2 \ll M^2)$ is an intrinsic observable of the CFT – it is well defined modulo an integer.

- $\kappa_{IR} = \lim_{p \to 0} \kappa(p^2)$ must be quantized.

Currents in $\mathcal{N} = 2$ Supersymmetry in $3d$

Supersymmetrizing the previous discussion, we distinguish between ordinary global symmetries and an R-symmetry.

A global non-R $U(1)$ symmetry can be coupled to a classical background $U(1)$ gauge superfield: $(a_\mu, \sigma, D, \lambda_\alpha)$.
(None of them including the "auxiliary field" D are constrained to satisfy their equations of motion.)

The supersymmetric CS term is

$$\frac{\kappa}{4\pi} \left(i\epsilon^{\mu\nu\rho} a_\mu \partial_\nu a_\rho - 2\sigma D + \text{fermions} \right) .$$

Supersymmetry relates the contact term in two currents to a contact term between two scalar operators.

Again, $\kappa \bmod(1)$ is physical.

R-Symmetry

The $U(1)_R$ current is in the same supermultiplet as the energy momentum tensor and the supersymmetry current.

The classical background fields are the metric $g_{\mu\nu}$, a complex gravitino, a $U(1)_R$ gauge field A_μ, a scalar H and a conserved vector V_μ.

In ordinary (new-minimal) $3d$ supergravity A_μ, H and V_μ are auxiliary fields – determined by their equations of motion. Here they are arbitrary background fields.

The relevant Chern-Simons contact terms are

- Flavor-R: $a \wedge dA + \cdots$
- Lorentz: $\omega \wedge d\omega + \cdots$
- R-R: $A \wedge dA + \cdots$

Flavor-R Chern-Simons Term

The contact term in the two-point function of the R-current and a non-R flavor current is related by supersymmetry to a number of other contact terms. These are summarized by the supersymmetric local counterterm

$$i\varepsilon^{\mu\nu\rho}a_\mu\partial_\nu\big(A_\rho - \frac{1}{2}V_\rho\big) + \frac{1}{4}\sigma R - DH + \cdots + \text{fermions}$$

This expression violates conformal invariance.

- ▶ The auxiliary fields $A_\mu - \frac{1}{2}V_\mu$ and H and the Ricci scalar R are not gauge invariant in conformal supergravity.
- ▶ These three fields couple to redundant operators in a CFT; e.g. R couples to T^μ_μ.

Lorentz Chern-Simons Term

The Lorentz Chern-Simons term can be supersymmetrized

$$\frac{i}{4}\varepsilon^{\mu\nu\rho}\,\mathrm{Tr}\left(\omega_\mu\partial_\nu\omega_\rho+\frac{2}{3}\omega_\mu\omega_\nu\omega_\rho\right)+i\varepsilon^{\mu\nu\rho}\left(A_\mu-\frac{3}{2}V_\mu\right)\partial_\nu\left(A_\rho-\frac{3}{2}V_\rho\right)$$

$$+\cdots+\mathrm{fermions}$$

These terms are superconformal.

R-R Chern-Simons Term

There is another independent supersymmetric counterterm

$$i\varepsilon^{\mu\nu\rho}\big(A_\mu - \frac{1}{2}V_\mu\big)\partial_\nu\big(A_\rho - \frac{1}{2}V_\rho\big) + \frac{1}{2}HR + \cdots + \text{fermions}$$

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Summary of the previous discussion

We studied four kinds of CS terms

- Flavor-flavor: $a \wedge da$
- Flavor-R: $a \wedge dA$
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They are conformal.

The supersymmetrization of the flavor-flavor and the Lorentz CS terms leads to superconformal expressions.

The supersymmetrization of the flavor-R and the R-R CS terms are not superconformal.

A New Anomaly

The flavor-flavor and the Lorentz Chern-Simons terms can be completed to superconformal expressions.

Therefore, the corresponding contact terms can be nonzero in a superconformal theory.

As above, their fractional parts are physical.

The flavor-R and the R-R Chern-Simons terms cannot be completed to superconformal expressions. (The expressions above are supersymmetric but not conformal.)

What should we do about these nonconformal terms?

A New Anomaly

Given a superconformal field theory we would like to impose:

- ▶ Supersymmetry
- ▶ Conformal symmetry
- ▶ All flavor and R-symmetries are compact

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The most conservative approach is to cancel the contact terms by adding CS terms with fractional coefficients.
(This is similar to the known framing anomaly [Witten].)

Then, the functional integral is not fully gauge invariant – it can change by a phase.

More precisely, defining the CS terms by extending the fields to $4d$, the results depend on the extension.

Example: $\mathcal{N} = 2$ SQED

This is a $U(1)$ gauge theory with N_f flavors and a CS level k.

Explicit computations uncover nonzero CS contact terms (for background fields) in the IR CFT.

- Flavor-flavor 2-pt function leads to $\kappa^{ff} = \frac{\pi^2 N_f}{4k} + \mathcal{O}(\frac{1}{k^3})$.
- Flavor-R 2-pt function leads to $\kappa^{fR} = -\frac{N_f}{2k} + \mathcal{O}(\frac{1}{k^3})$.
- We expect the R-R and the Lorentz terms to be nonzero.

The flavor-R and the R-R contact terms violate the superconformal symmetry of the IR theory.

We can cancel them by adding appropriate counterterms, violating invariance under large gauge transformations of the background fields.

Placing an $\mathcal{N} = 2$ Theory on S^3

An $\mathcal{N} = 2$ theory with a $U(1)_R$ symmetry can be placed on S^3, while preserving supersymmetry [D. Sen; Romelsberger; Kapustin, Willett, Yaakov].

We follow [Festuccia, NS] and turn on background superfields:

- We need to turn on $H = -\frac{i}{r}$. Here H is a scalar background superpartner of the metric and r is the radius.
- For any non-R $U(1)$ symmetry we can add a background gauge superfield with $D = \frac{i\sigma}{r}$ with complex σ.
- $\mathrm{Re}(\sigma) =$ real mass term. $\mathrm{Im}(\sigma)$ represents a choice of an R-symmetry [Jafferis; Hama, Hosomichi, Lee].
- The background fields H, σ and D do not have their standard reality – a possible problem with unitarity.

The Sphere Partition Function

- The sphere partition function $Z = e^{-F}$ of a unitary field theory should be real (even if it is not parity invariant). This follows from reflection positivity.

- If we want to preserve supersymmetry, we need to turn on imaginary H, which violates reflection positivity. Hence, the partition function could be complex.

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▶ This should not be an issue in a superconformal theory, where H decouples.

▶ Nevertheless, explicit calculations based on localization exhibit complex answers for SCFTs on S^3.

The Phase of the S^3 Partition Function

Starting in flat space, we find the four supersymmetric Chern-Simons contact terms. Two of them are not superconformal.

If we do not add bare counterterms to remove them, the superconformal field theory has nonconformal contact terms.

Substituting the complex background values of H, σ and D in these terms we find a nontrivial phase of Z (violating reflection positivity).

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The phase of the partition function is thus computable using the flat-space values of the contact terms. It agrees with the localization computations on S^3.

The anomaly discussed above is seen now as a clash between unitarity and full background gauge invariance.

F-maximization

Consider an SCFT with some flavor symmetries.
Explore the sphere partition function $Z = e^{-F}$ as a function of
background gauge superfields for these symmetries $(\sigma, a_\mu, ...)$.
The parameter $t = \text{Im}(\sigma)$ shifts the choice of R-symmetry in
the superconformal algebra.

We add local counterterms to restore the superconformal
symmetry and unitarity.

- They are incompatible with the full background gauge
  invariance.
- They set $\text{Im}(F)$ to zero and shift $\text{Re}(F)$.

Then,

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Hence, F is at a maximum (conjectured by [Jafferis, Klebanov, Pufu, Safdi]).

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- The (flat space) observables $\kappa \bmod(1)$ and τ are calculable using localization on the sphere.

- They are independent of superpotential couplings including exactly marginal deformations.

- F-maximization is closely related to the "F-theorem."

Tests of Dualities

There are several conjectured dualities between different $3d$ theories; e.g. $3d$ mirror symmetry, Aharony duality, Giveon-Kutasov duality, etc.

They can be tested by comparing the S^3 or the $S^2 \times S^1$ partition functions of the two dual theories [Kapustin, Willett, Yaakov; Benini, Closset, Cremonesi; ...].

The results almost agree.

- They can be made to agree by adding to one of the sides of the duality CS counterterms for the background fields. (Equivalently, these are corrections to the definition of the global symmetry currents.)
- Following our discussion, these counterterms should have quantized coefficients.
  (Generalization of matching the parity anomaly in dual $3d$ theories [Aharony, Hanany, Intriligator, NS, Strassler].)
- Furthermore, their quantized coefficients can be determined independently.
  They can be calculated at one loop on R^3 by comparing different dual pairs, which are related by renormalization group flows.

Example

One-loop flat-space computations in Giveon-Kutasov duality
($U(N_c)_k$ with N_f flavors) predict the correction term

$$\delta F = \pi \left[iN_f(N_f - k)m^2 + i\xi^2 + N_f(N_f + k - 2N_c)m \right] + \cdots ,$$

where m is a real mass and ξ is an FI-parameter.
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The agreement is a nontrivial test of:
- ▶ The dualities
- ▶ Our entire understanding of these contact terms

Conclusions

- Contact terms are usually arbitrary.
- Chern-Simons contact terms lead to new computable observables.
- The natural way to describe them is in terms of counterterms of background gauge and (super)gravity (super)fields.
- Some Chern-Simons contact terms are not superconformal – like an anomaly.
- In order to preserve supersymmetry on curved space, we should turn on various supergravity background fields.

- The non-conformal Chern-Simons terms lead to a phase of the S^3 partition function (violation of unitarity) even for conformal theories.
- Removing these terms (by tentatively sacrificing full background gauge invariance) we proved the conjectured F-maximization principle.
- This understanding leads to new non-trivial tests of conjectured dualities.