Fermion Quantum Numbers in Kaluza-Klein Theory

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The problem of obtaining left-right asymmetry of fermion quantum numbers in Kaluza-Klein theory is discussed. In the absence of elementary gauge fields, a theorem by Atiyah and Hirzebruch states that the Dirac equation in $4 + n$ dimensions always leads to vectorlike fermion quantum numbers in 4 dimensions. The proof of this theorem is sketched. It is shown that the same holds for the Rarita-Schwinger operator *on homogeneous spaces*, but a *general* impossibility theorem for the Rarita-Schwinger field is not proved. (However, in view of the apparent restriction of supergravity to $d \leqslant 11$, this line of approach is severely constrained.) Also discussed are some Kaluza-Klein theories with elementary gauge fields, some difficulties in obtaining massless charged scalars, and some speculations about the cosmological constant.

I Introduction

Kaluza-Klein theory [1] has recently attracted increasing interest as a program for unifying gauge interactions with gravity. This theory can be viewed [2] in terms of spontaneous symmetry breaking, the compact and noncompact dimensions being on an equal footing as far as the laws of nature are concerned, just as the photon and the massive vector mesons are treated symmetrically in the standard weak interaction models. This simple observation is one of the chief reasons for the revived interest [3] in Kaluza-Klein theory.

One may start with a general relativistic theory in $4 + n$ dimensions and assume the grand state to be $M^4 \times B$, where M^4 is 4-dimensional Minkowski space and B is a compact space. Continuous symmetries of B[1] will always be observed [4] as gauge symmetries in the effective 4-dimensional world. The gauge fields (which in general can be more numerous than the extra dimensions) originate in the normal mode expansion of the fluctuations in the $(4 + n)$-dimensional metric tensor. For instance, starting in 11 or more dimensions, one can [4, 5] obtain gauge fields of $SU(3) \times SU(2) \times U(1)$. Much of this paper will be devoted to the consequences of assuming that all observed gauge forces originate in this way, as part of the metric tensor, from a theory that originally had no elementary gauge fields. However, this is not necessarily the only attractive possibility. It might be equally attractive to start with a unified theory (perhaps a supergravity [6]

1. That is transformations that leave fixed both the geometry of B and the expectation values of any matter fields that may be present.

or superstring [7] theory) that determines the original gauge group. [For instance, one of the $n = 2$ supergravity theories in 10 dimensions requires the existence of a $U(1)$ gauge field.] Our remarks in section VI will be relevant to such theories.

As soon as one begins to think about Kaluza-Klein theory, one faces a bewildering variety of choices. There are many assumptions one might make, and many facts about elementary particle physics one might try to explain. It appears unlikely that at the present time we can guess correctly the whole detailed form of the $(4 + n)$-dimensional laws and all the key points of the dynamics. For these reasons, it seems important to isolate problems that can be addressed without claiming to understand all the details of a theory. As will become apparent, the problem of trying to predict the quark and lepton quantum number is such a problem, and it will be our main interest in this paper. However, we shall also make some remarks on certain other qualitative problems: the gauge hierarchy problem and the problem of the cosmological constant.

Since we shall deal mainly with the problem of the fermion quantum numbers, it is worthwhile to recall briefly some aspects of that problem as it presently appears.

One of the most striking aspects of particle physics is that left-handed fermions transform under $S(3) \times SU(2) \times U(1)$ differently from the way the right-handed fermions transform. (The quantum numbers are not "vectorlike.")

For instance, left-handed quarks are $SU(2)$ doublets, but right-handed quarks are $SU(2)$ singlets. Equivalently, one may say that the fermions of given helicity form a complex representation of $SU(3) \times SU(2) \times U(1)$. The fermions of one generation transform under $SU(3) \times SU(2) \times U(1)$ as $(3,2)^{1/3} \oplus (\bar{3},1)^{-4/3} \oplus (\bar{3},1)^{2/3} \oplus (1,1)^{2} \oplus (1,2)^{-1}$, which is a so-called complex representation. [In other words, it is not equivalent to its complex conjugate, which is $(3,2)^{-1/3} \oplus (3,1)^{4/3} \oplus (3,1)^{-2/3} \oplus (1,1)^{-2} \oplus (1,2)^{1}$; by *CPT*, this is the representation furnished by the *right-handed* fermions.] This fact is of utmost importance, because it means that bare masses of the quarks and leptons are forbidden by gauge invariance. The quarks and leptons can acquire mass only when $SU(2) \times U(1)$ is spontaneously broken. This, in turn, means that the quarks and leptons cannot be much heavier than the mass scale at which $SU(2) \times U(1)$ is broken; they cannot have masses of order, say, the Planck mass. The relative "lightness" of the fermions would therefore be explained if the "smallness" of the $SU(2) \times$

$U(1)$ breaking scale were understood; it is not an independent problem. In this paper we shall assume, in accord with the "survival hypothesis" [8], that the only light fermions are fermions that are required to be light by gauge invariance; this assumption will not always be explicitly stated.

The fact that the quantum numbers are not vectorlike means that the spectrum of light fermions[2] depends only on the "universality class" of an $SU(3) \times SU(2) \times U(1)$ invariant theory. The lightness of the light fermions and their quantum numbers cannot be modified by any $SU(3) \times SU(2) \times U(1)$ invariant perturbations. We do not know at what length scale the spectrum of light fermions is determined, but it may be that this reflects physics at the smallest length scales.

Of course, there is no experimental *proof* that mirror fermions with $V + A$ couplings to the usual W mesons will not be found, restoring the vectorlike nature of the fermion spectrum. But there are many reasons to doubt that this will occur. If mirror fermions are discovered, we shall lose our theoretical understanding of why the quarks and leptons are $\lesssim M_W$ in mass. (Of course, in any case we don't understand why some of the fermions are so much *lighter* than M_W.) If mirror fermions do exist, they fail to a remarkable extent to mix with the usual fermions; the first-generation fermions are very light and have almost pure $V - A$ weak interactions. If mirrors do exist, it is odd that none of the 14 $SU(3) \times SU(2) \times U(1)$ multiplets observed so far appear to be mirrors. This is all the more remarkable in that the mirrors cannot weigh more than at most a few hundred GeV. [Since they do not mix with usual fermions, their bare masses are forbidden by $SU(2) \times U(1)$.] Finally, and perhaps most convincingly, the triangle anomalies cancel among the observed quarks and leptons. This cancellation appears to be a rather striking confirmation of current ideas, but if mirrors exist, it is just an elaborate and unnecessary charade, since the mirrors would automatically cancel the anomalies of the known fermions, whatever those anomalies might be.

What is more, the fermion representation is complicated [each family consits of five irreducible representations of $SU(3) \times SU(2) \times U(1)$] and redundant (there are three families). On the first point, no doubt the $SU(5)$ and $O(10)$ grand unified theories are the most successful efforts to date. [A

2. By the "spectrum of light fermions" we mean the quantum numbers of the light fermions. We shall usually speak of the quarks and leptons as if they were massless, ignoring the $SU(2) \times U(1)$ breaking.

family is $\bar{5}_L + 10_L$ in $SU(5)$, or 16_L in $O(10)$.] On the second point—which is an updated version of Rabi's question, "Who ordered the muon?"—there is no equally convincing answer. It is natural to try to embed the three families as one irreducible representation of a bigger group. Perhaps the most attractive such idea is to use the spinor representation [9] of $O(N)$ for $N \geqslant 18$. The spinor representation of $O(N)$ is the representation space of N gamma matrices, which automatically furnishes a representation of any subset of the gamma matrices; so the $O(N)$ spinor transforms as a sum of spinor representations of any minimally embedded $O(k)$ subgroup. For instance, the irreducible spinor of $O(18)$ transforms under $O(10)$ as four families plus (unfortunately) four antifamilies. This beautifully achieves the desired multiplicity, but it is not easy to eliminate the antifamilies. One may invoke a "hypercolor" force that becomes strong at ~ 1 Tev, breaking $SU(2) \times U(1)$ and confining the antifamilies. This elegant idea [10] has innumerable difficulties in detail. In this paper, difficulties will be much more conspicuous than phenomenological successes, but we shall note, in section VI, that Kaluza-Klein theory gives an alternative way of avoiding antifamilies in the $O(N)$ approach to the family problem.

A discussion of zero modes of nontrivial Dirac operators in Kaluza-Klein theory was apparently first given in a special situation by Palla [11]. In reference [4], in connection with a discussion of some pseudorealistic models, the detailed proposal was made that the quark and lepton quantum numbers are determined by the topology of a manifold with $SU(3) \times SU(2) \times U(1)$ symmetry. The importance and difficulty of obtaining a complex representation were pointed out. Chapline and Slansky and Manton [12] discussed the problem of obtaining a complex spectrum in Kaluza-Klein theory; they anticipated the kinematical analysis of section II and some of the ideas of section VI. The kinematical analysis has been recently developed in much more detail by Wetterich [13], who worked out all of the kinematical consequences of the mod 8 periodicity of the spinor representation of $O(N)$. He also introduced in a different language the mathematical concept of the character-valued index, which, as we shall see, plays a very important role. Models exhibiting many of the ideas of section VI have been analyzed by Randjbar-Daemi, Salam, and Strathdee [14].[3] As regards

3. I understand, in addition, that these authors have considered (unpublished) some of the detailed models in section VI.

fermion quantum numbers, the novelty in the present paper is primarily the restrictions disussed in sections IV and V and the more realistic models in section VI.

Our conclusions will be as follows. If all gauge fields are part of the metric tensor, then a theorem of Atiyah and Hirzebruch [15] states that the Dirac operator in 4 + n dimensions always leads to vectorlike quantum numbers in 4 dimensions. (The relevance of this theorem to Kaluza-Klein theory was first noted in reference [16].) For the Rarita-Schwinger operator the situation is more complicated. We shall show that if the hidden dimensions form a homogeneous space, the Rarita-Schwinger operator likewise always leads to vectorlike quantum numbers. What happens in general for the Rarita-Schwinger operator on spaces that are not homogeneous I do not know. However, the fact that supergravity is apparently restricted to $d \leqslant 11$ [17] and certain other facts discussed in section V indicate that this avenue is not promising. If one is less ambitious and introduces elementary gauge fields in $4 + n$ dimensions, it is possible, but still subtle, to get complex representations. Indeed, as we shall see in section VI, one can naturally get very big, complicated, duplicated representations. In section VII, we discuss some other ways that the assumptions might be modified.

We shall encounter considerable difficulties in our attempts to interpret the fermion quantum numbers as the solution of an index problem. Nevertheless, this seems to be a quite attractive idea.

II Preliminaries

Let us first recall how—in a Kaluza-Klein theory with ground state $M^4 \times B$—massless particles originate as zero modes of appropriate wave operators on B. A massless Dirac particle in $4 + n$ dimensions obeys

$$0 = \not{D}\psi = \sum_{i=1}^{4+n} \Gamma^\mu D_\mu \psi, \tag{1}$$

where Γ^μ, $i = 1, \ldots, 4 + n$, are the gamma matrices. Notice that we may as well use the minimal Dirac equation. Even if nonminimal terms (couplings to matter fields or nonminimal couplings to gravity) are present, they cannot change the quantum numbers of massless fermions in a complex representation of the symmetry group. This is an illustration of the fact that the problem of fermion quantum numbers depends only on the "universality class" of a theory. We do not have to believe we know which Dirac operator is physically relevant. If we define the 4-dimensional Dirac operator $D\!\!\!/^{(4)} = \sum_{\mu=1}^{4} \Gamma^\mu D_\mu$ and the internal Dirac operator $D\!\!\!/^{(n)} = \sum_{j=5}^{4+n} \Gamma^j D_j$, then (1) becomes

$$0 = D\!\!\!/^{(4)}\psi + D\!\!\!/^{(n)}\psi. \tag{2}$$

We see that $D\!\!\!/^{(n)}$ is the mass operator, in effect. Its eigenvalues are observed in 4 dimensions as the particle masses. Its zero eigenvalues are the massless fermions.

The Rarita-Schwinger operator may be discussed similarly. There are many, equivalent ways [6] to write the $(4+n)$-dimensional Rarita-Schwinger equation. One way is

$$0 = \Gamma^\mu(D_\mu\psi_\nu - D_\nu\psi_\mu), \tag{3}$$

where $\psi_{\mu\alpha}$ is the Rarita-Schwinger field ($\mu = 1, \ldots, 4+n$ is a vector index; α is a spinor index). In the gauge $\Gamma^\mu\psi_\mu = 0$, (3) reduces to

$$0 = \Gamma^\mu D_\mu\psi_\nu = D\!\!\!/^{(4)}\psi_\nu + D\!\!\!/^{(n)}\psi_\nu. \tag{4}$$

Again zero modes of $D\!\!\!/^{(n)}$ are observed as massless particles in 4 dimensions. The general zero mode is a sum of modes of two special kinds. For $\nu = 1, \ldots, 4$, $D\!\!\!/^{(n)}$ is the ordinary spin $\frac{1}{2}$ Dirac operator, and the zero modes are spin $\frac{3}{2}$ fermions in 4 dimensions. For $\nu = 5, \ldots, 4+n$, the zero modes have spin $\frac{1}{2}$ as seen in 4 dimensions, while their dependence on the compact dimensions is determined by the gauge condition *and* the Dirac-like equation

$$0 = \sum_{\nu=5}^{4+n} \Gamma^\nu\psi_\nu, \qquad 0 = \sum_{\mu=5}^{4+n} \Gamma^\mu D_\mu\psi_\nu. \tag{5}$$

These conditions, taken together, are equivalent to the gauge invariant internal Rarita-Schwinger equation

$$\sum_{\mu=5}^{4+n} \Gamma^\mu(D_\mu\psi_\nu - D_\nu\psi_\mu) = 0, \qquad \nu = 5, \ldots, 4+n, \tag{6}$$

in a particular gauge. (We temporarily introduced a gauge fixing condition only to decouple the Minkowski dimensions from the compact ones.) We see that zero modes of the internal Rarita-Schwinger operator become massless spin $\frac{1}{2}$ fermions in 4 dimensions.

Now, can either of these operators have zero eigenvalues? And can the zero eigenvalues form complex representations of a symmetry group?

The crudest problem, which was pointed out in (4), arises in an *odd* number of dimensions. For odd n, the group $O(n)$ only has one spinor representation. Likewise, the group $O(1,3+n)$ has only one spinor representation, which transforms under $O(1,3) \times O(n)$ as the product of the four component spinor of $O(1,3)$ with the unique spinor of $O(n)$. This being so, fermions that are left- or right-handed in 4 dimensions transform the same way under transformations of the internal space. They obey the same Dirac equation in the internal space (modulo nonminimal terms that cannot affect the quantum numbers), so they have the same quantum numbers and furnish a real representation of any relevant symmetry group.

In an even number of dimensions the situation is more subtle. For even n the operator $\hat{\Gamma} = \Gamma^1 \cdots \Gamma^n$ anticommutes with all Γ^i, so it is a c-number in any representation of the Clifford algebra. Since $\hat{\Gamma}^2 = \pm 1$ (depending on n), the representation space of the Clifford algebra decomposes into two eigenspaces of $\hat{\Gamma}$, the eigenvalues being ± 1 or $\pm i$. Since $\hat{\Gamma}$ commutes with the $O(n)$ group generators $\frac{1}{4}[\Gamma^i, \Gamma^j]$, the group has two inequivalent spinor representations, labeled by the eigenvalue of $\hat{\Gamma}$.[4]

In a world of $4+n$ dimensions we define

$$\begin{aligned} \bar{\Gamma} &= \Gamma^1\Gamma^2 \cdots \Gamma^{4+n}, \\ \Gamma^{(4)} &= \Gamma^1\Gamma^2 \cdots \Gamma^4, \\ \Gamma^{\mathrm{Int}} &= \Gamma^5\Gamma^6 \cdots \Gamma^{4+n}. \end{aligned} \tag{7}$$

These operators have simple interpretations. $\bar{\Gamma}$ labels the spinor representations of $O(1,3+n)$, $\Gamma^{(4)}$ measures the helicity of 4-dimensional fermions, and Γ^{Int} labels the spinor representations of $O(n)$; it measures what might be called the internal helicity. These operators obey the simple relation

$$\bar{\Gamma} = \Gamma^{(4)} \cdot \Gamma^{\mathrm{Int}}. \tag{8}$$

This equation has an important consequence. For fixed $\bar{\Gamma}$, the 4-dimensional

4. For odd n, $\hat{\Gamma}$ commutes with the Γ^i and is a c-number in an irreducible representation of the Clifford algebra. The Clifford algebra thus has two inequivalent representations, labeled by the sign of $\hat{\Gamma}$. They are related to each other by $\Gamma^i \to -\Gamma^i$ (which for odd n yields $\hat{\Gamma} \to -\hat{\Gamma}$), and they are equivalent as representations of $O(n)$ since under $\Gamma^i \to -\Gamma^i$ the group generators $\frac{1}{4}[\Gamma^i, \Gamma^j]$ are unchanged.

and internal chiralities are correlated. If we start with a fermion field restricted to (say) $\bar{\Gamma} = 1$ in $4 + n$ dimensions, it breaks down under $O(1,3) \times O(n)$ to components with[5]

$$\eta\Gamma^{(4)} = \eta^{-1}\Gamma^{\text{Int}} = +1 \qquad \text{or} \qquad \eta\Gamma^{(4)} = \eta^{-1}\Gamma^{\text{Int}} = -1. \tag{9}$$

Fermions of left- or right-handed physical helicity are left- or right-handed in the internal space. They obey different Dirac equations, whose zero modes might have different quantum numbers.

This idea quickly runs into trouble if the number of dimensions is divisible by four. In $4k$ dimensions, $\bar{\Gamma}$ is odd under CPT. This may be seen readily in a Majorana basis, with real gamma matrices. In such a basis CPT acts on spinors just by complex conjugation. $\bar{\Gamma}$ [as defined in (7)] is real in a Majorana basis, but in $4k$ dimensions it may be readily seen to obey $(\bar{\Gamma})^2 = -1$. The eigenvalues of $\bar{\Gamma}$ are $\pm i$. Being complex conjugates, the eigenvalues of $\bar{\Gamma}$ are related to each other by CPT, and CPT requires that there be equal numbers of fields with $\bar{\Gamma} = +i$ and $\bar{\Gamma} = -i$. Hence there is no net correlation between 4-dimensional chirality and internal chirality. Fields of $\bar{\Gamma} = +i$ give one correlation and fields of $\bar{\Gamma} = -i$ give the opposite correlation. Thus, in $4k$ dimensions, CPT requires that the gravitational interactions be vectorlike. Naturally, therefore, if the weak interactions are part of the gravitational force in $4 + n$ dimensions, the weak interactions are also vectorlike. Alternatively, one may say [13, 18] that in $4k$ dimensions, bare masses are possible for any fermions coupled to gravity only. Such a theory will, of course, always reduce to an appropriate 4-dimensional theory in which bare masses are still possible.

In $4k + 2$ dimensions, the situation is very different. In this case $\bar{\Gamma}^2 = +1$, so $\bar{\Gamma}$ has eigenvalues ± 1. CPT leaves $\bar{\Gamma}$ unchanged, and we can consider a theory with fermions of (say) $\bar{\Gamma} = +1$ only. (This option is forced on us in certain situations—for instance, in certain 10-dimensional supersymmetric field theories and string theories.) This corresponds roughly to a theory with $V - A$ gravitational interactions that forbid fermion bare masses; the question is whether $V - A$ gravity can reduce to $V - A$ weak interactions in 4 dimensions.

The special role of $4k + 2$ dimensions in multidimensional field theory was first raised in constructing supersymmetric Yang-Mills theories [19].

5. Here η is a phase factor that will be determined momentarily. It can be ignored for the time being.

In the context of analyzing fermion quantum numbers this point was made and developed in references [13] and [14]. Similar observations are important in grand unified model building [9]. In the mathematical literature the periodicity of the spinor representation is an old observation [20].

Let us now work out the phase factor η of equation (9). We note that as defined $\Gamma^{(4)}$ is (in a Majorana basis) a real matrix whose square is -1, so the eigenvalues of $\Gamma^{(4)}$ are $\pm i$. In $4k+2$ dimensions Γ^{Int} likewise has square -1 and eigenvalues $\pm i$. [In $4k$ dimensions $(\Gamma^{\text{Int}})^2 = +1$.] A fermi field that obeys $\bar{\Gamma} = \Gamma^{(4)}.\Gamma^{\text{Int}} = +1$ therefore has

$$\text{(A)}\quad \Gamma^{(4)} = -\Gamma^{\text{Int}} = +i \qquad \text{or} \qquad \text{(B)}\quad \Gamma^{(4)} = -\Gamma^{\text{Int}} = -i. \tag{10}$$

A CPT transformation will complex conjugate the eigenvalues, so eigenvalues of type A and B are exchanged by CPT. This is as it should be. A zero mode of the internal Dirac or Rarita-Schwinger operator with $\Gamma^{\text{Int}} = -i$ corresponds to a left-handed massless fermion in 4 dimensions. Its complex conjugate will have $\Gamma^{\text{Int}} = +i$ and corresponds to a right-handed massless fermion in 4 dimensions. Massless fermions in 4 dimensions will transform in a complex representation of some symmetry group G if the zero modes of the internal Dirac operator with $\Gamma^{\text{Int}} = -i$ form a complex representation of G, or equivalently, if the zero modes of $\Gamma^{\text{Int}} = +i$ transform differently from those of $\Gamma^{\text{Int}} = -i$.

Since the remainder of this section and the next one will deal with "chiral theories of gravity" with elementary fermi fields of a definite value of $\bar{\Gamma}$, it should be mentioned at the outset that these theories suffer from a major problem. The fermion one-loop diagrams in an external gravitational field are anomalous [18]. (The anomaly first appears in a diagram with $2k+2$ external gravitons.) In a few special theories in 6 or 10 dimensions the anomalies cancel between fields of different spin; beyond 10 dimensions this is impossible. We shall not limit ourselves to the anomaly-free theories but will investigate the fermion quantum numbers that emerge (at the tree level) in the whole class of chiral gravity theories. There are several justifications for ignoring the anomalies. First, general methods for treating the whole class of chiral gravity theories are as simple as any special methods for analyzing the particular anomaly-free theories. And the general methods are likely to be important for other attempts to calculate fermion quantum numbers in Kaluza-Klein theory; for instance, we shall apply them from a different standpoint in section VI. Second, it may happen that in the future a massless field of some exotic spin might be successfully coupled to gravity.

This could expand the room for cancellation of anomalies without affecting our tree level considerations for spin $\frac{1}{2}$ and spin $\frac{3}{2}$ fields. Third, though it seems unlikely, perhaps there is some way to make sense of anomalous field theories or of other theories whose low energy limit is an anomalous field theory.

A rather simple argument due essentially to Lichnerowicz [21] severely limits the possibility of obtaining zero modes of the Dirac operator in complex representations of a symmetry group. If one squares the internal Dirac operator (which will simply be denoted $i\not{D}$; we henceforth suppress the 4 Minkowskian dimensions), we find

$$\begin{aligned}(i\not{D})^2 &= -\sum D_i D^i - \tfrac{1}{4}[\gamma^i,\gamma^j][D_i,D_j] \\ &= -\sum D_i D^i - \tfrac{1}{32}[\gamma^i,\gamma^j][\gamma^k,\gamma^l]R_{ijkl} \\ &= -\sum D_i D^i + \tfrac{1}{4}R.\end{aligned} \tag{11}$$

Since $-\sum D_i D^i$ is a nonnegative operator, this shows that if $R > 0$ everywhere, the Dirac operator has no zero eigenvalues. Of course, in (11) we have considered a minimally coupled Dirac operator. If nonminimal couplings are present, the Dirac operator may have zero eigenvalues. But the fact that the minimally coupled operator has no zero eigenvalues at all, and leads to no massless fermions in 4 dimensions, means that even in the presence of nonminimal couplings, the zero eigenvalues form real representations of whatever symmetry group may be present.[6] A particularly important case of this is the following. Suppose the compact space B has a symmetry group G. In general there will be many G-invariant metrics on B. If even one of them has $R > 0$, then for *any* G-invariant metric on B, the zero modes of the Dirac operator, if any, form a real representation of G. (Of course, under these circumstances the Dirac operator will generically have no zero eigenvalues.)

Much of the literature on Kaluza-Klein theory has concerned homogeneous spaces $B = G/H$, G and H being compact nonabelian groups. These spaces all admit a canonical G-invariant metric of positive scalar curvature, so (even if nonminimal terms are added or a different G-invariant metric is used) they give real representations for zero eigenvalue. More generally, a theorem by Lawson and Yau [22] shows that on any compact space B (not necessarily a homogeneous space) with a nonabelian symmetry G, there is a

6. This should be obvious "physically" from the connection with fermion quantum numbers in 4 dimensions. The precise mathematical argument will be given shortly.

G-invariant metric of positive scalar curvature R. For nonabelian groups such as $SU(3) \times SU(2) \times U(1)$, this rules out the possibility of getting zero modes of the Dirac operator in complex representations.

This simple line of argument does not address the question (of conceptual but probably not of practical interest) whether zero modes of the Dirac operator can form a complex representation of an abelian symmetry group. (Manifolds with a continuous abelian symmetry group in general do not admit an invariant metric of positive scalar curvature.) Much more important, this line of reasoning does not extend to the Rarita-Schwinger operator whose square is more complicated than (11) and is not manifestly positive even if $R > 0$. For this reason, the Rarita-Schwinger operator can have zero eigenvalues more readily than the Dirac operator. For instance, in 4 dimensions there is one compact manifold that is not flat but obeys $R_{\mu\nu} = 0$. It is the K3 surface, and for topological reasons it has two zero eigenvalues of the Dirac operator and 42 zero eigenvalues of the Rarita-Schwinger operator.

In multidimensional supergravity and superstring theories—which are the only known theories in which fermions are really unified with gravity—we inevitably are dealing with Rarita-Schwinger fields. It therefore is important to learn to analyze the zero modes of these fields.

There is another no-go theorem, due to Atiyah and Hirzebruch [15], which for our particular problem is much more restrictive than the reasoning just sketched. They proved precisely that for *any* continuous symmetry group, abelian or nonabelian, the Dirac zero modes form a real representation. As we shall see, their argument has important implications for the Rarita-Schwinger case; for instance, we shall use it to prove that if the compact space is a homogeneous manifold, the Rarita-Schwinger operator always leads to a real representation. We shall present in section IV an elementary proof of the Atiyah-Hirzebruch theorem that is closely related to the original argument.

Why would a wave operator have zero eigenvalues? And why would these zero eigenvalues form complex representations? We shall illustrate the relevant concepts in terms of the Dirac operator. In the rest of this paper, we suppress the 4 Minkowskian dimensions and concentrate on properties of the n-dimensional Kaluza-Klein space B. To streamline notation, gamma matrices are henceforth gamma matrices $\Gamma^1, \Gamma^2, \ldots, \Gamma^n$ of B, and we define an operator $\hat{\Gamma} = i\Gamma^{\mathrm{Int}} = i\Gamma^1 \cdots \Gamma^n$ with eigenvalues ± 1. Indices i, j, k refer to the internal space; indices μ, ν, α refer to all $4 + n$ dimensions.

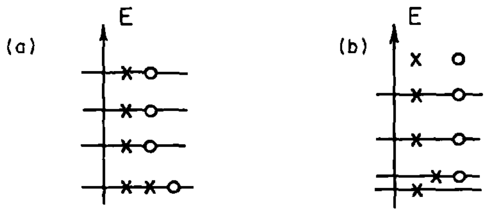

Figure 1
Zero modes of the Dirac operator of positive or negative chirality are indicated by × or ○, respectively. The number of ×s minus ○s at zero energy is invariant under perturbations.

Let us define a "Hamiltonian" $H = (i\not{D})^2$. Since $[\hat{\Gamma}, H] = 0$, H eigenstates can be chosen to be at the same time $\hat{\Gamma}$ eigenstates. If $H\psi = E\psi$, then $H \cdot i\not{D}\psi = E \cdot i\not{D}\psi$. So ψ and $i\not{D}\psi$ are degenerate in energy, unless $i\not{D}\psi = 0$. But since $\not{D}\hat{\Gamma} = -\hat{\Gamma}\not{D}$, ψ and $i\not{D}\psi$ have opposite eigenvalues of $\hat{\Gamma}$. Consequently (figure 1), the H eigenvalues of nonzero energy are paired. For every state of $\hat{\Gamma} = 1$ there is a state of $\hat{\Gamma} = -1$. The zero eigenvalues need not be paired in this way. The number of zero eigenvalues of $\not{D}$ with $\hat{\Gamma} = 1$ minus the number with $\hat{\Gamma} = -1$ is called the *index* of $\not{D}$.

The index is invariant under arbitrary deformations of $\not{D}$ that preserve the property $\hat{\Gamma}\not{D} = -\not{D}\hat{\Gamma}$, since no smooth distortion of figure 1 that preserves the pairing at nonzero energy can disturb whatever chirality imbalance may exist at $E = 0$. In particular the index of $\not{D}$ is a topological invariant, depending on the topology of B but not on its metric tensor. Generically, in the absence of some symmetry principles (which can, however, change the situation, as we shall see), zero eigenvalues of $\not{D}$ all have positive chirality or all have negative chirality. This is so because zero eigenvalues of equal and opposite chirality would gain nonzero energy under a generical perturbation.

Although the index is the simplest deformation invariant of the Dirac operator that is relevant to the occurrence of zero modes, for our purposes we need a slightly different concept. In $4k + 2$ dimensions the index of the Dirac operator always vanishes, for the simple reason that the positive and negative chirality zero modes of the Dirac operator are complex conjugates of each other (as we have seen earlier) and therefore equal in number. The concept of interest to us is what mathematicians call the G-index or the character-valued index of the Dirac operator.

Let the manifold B have a symmetry group G. The eigenstates of H or $i\not{D}$

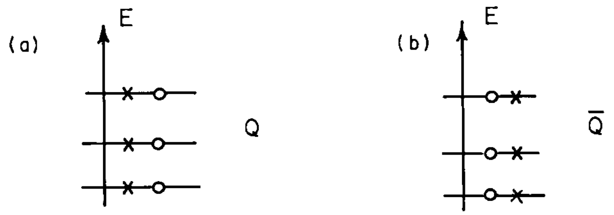

Figure 2
An × or ○ now indicates a positive or negative chirality multiplet in the Q or $\bar{Q}$ representation [(a) and (b), respectively]. In passing from (a) to (b) the ×s and ○s are exchanged.

will then form representations of G. Pick a representation Q, and draw the same picture as before (figure 2), but only counting multiplets in the Q representation. We define $\text{index}_Q(\not{D})$ to be the number of zero mode multiplets in the Q representation of positive chirality minus the number of zero mode multiplets in the Q representation of negative chirality. For reasons similar to those given earlier, $\text{index}_Q(\not{D})$ is invariant under arbitrary perturbations that respect G symmetry and preserve the fact that $\not{D}\hat{\Gamma} = -\hat{\Gamma}\not{D}$.

Of course, we can still complex conjugate our eigenstates of H. This still reverses the eigenvalue of $\hat{\Gamma}$, but now it exchanges Q with its complex conjugate representation $\bar{Q}$. By complex conjugation (figure 2b) this implies that $\text{index}_Q(i\not{D}) = -\text{index}_{\bar{Q}}(i\not{D})$. Upon reduction to 4 dimensions, this implies the perfectly valid statement that the number of left-handed massless fermions in the Q representation equals the number of right-handed massless fermions in the $\bar{Q}$ representation.

An equivalent way to define the character-valued index is as follows. The positive and negative chirality zero modes of $i\not{D}$ form representations Λ^+ and Λ^- of G. For $g \in G$ we define

$$\text{index}(g) = \text{tr}_{\Lambda^+}(g) - \text{tr}_{\Lambda^-}(g). \tag{12}$$

Or equally well we define

$$\text{index}(g) = \sum_Q \text{index}_Q(i\not{D})\,\chi_Q(g), \tag{13}$$

where $\chi_Q(g)$ is the trace of g in the Q representation.[7]

7. This definition of $\text{index}(g)$ makes sense in $4k$ as well as $4k + 2$ dimensions, though it is then not related to our physical problem, and we lose the identity $\text{index}(g) = (\text{index}(g^{-1}))^*$ that follows from complex conjugation in $4k + 2$ dimensions.

If the character-valued index is nonzero, $i\not{D}$ must have zero modes. Generically, the number of zero modes will be the minimum required to yield the right value of $\text{index}_Q(i\not{D})$ for each Q. The spectrum of zero modes required by the character-valued index we shall call the "stable spectrum" of zero modes. We shall usually assume that the actual spectrum of zèro modes coincides with the stable spectrum and can be computed by evaluating the character-valued index. From the fact that $SU(3) \times SU(2) \times U(1)$ forbids bare masses for all the known quarks and leptons, it appears that this assumption is valid in nature. A successful model would be one in which the character-valued index consists of three families minus three antifamilies.

Unfortunately, as we shall see in section IV, the Atiyah-Hirzebruch theorem ensures that the character-valued index always vanishes for the Dirac operator in theories without elementary gauge fields. At least on homogeneous spaces, this is also true for the Rarita-Schwinger field. It is not true (even on homogeneous spaces) for fields of spin $\frac{5}{2}$ or larger; we shall discuss a counterexample in section V (but there does not seem to be any way to use massless fields of spin $\geq \frac{5}{2}$ in physics [23]). The character-valued index also need not vanish in the presence of elementary gauge fields, and we shall construct some pseudorealistic models in section VI on the basis of this fact.

As we shall see, there are very powerful methods for calculating the character-valued index of arbitrary operators. It is never necessary to write down and solve an explicit differential equation.

III Operators on Homogeneous Spaces

For our first experience in calculating the character-valued index of various operators, we shall consider the simple case in which the manifold B is a homogeneous space G/H. In that case, there is a particularly elementary way [24] to compute the stable spectrum of zero modes of the Dirac operator (or any other G-invariant operator). One simply expands the spinor fields on B in harmonics (irreducible representations) of the group G. For any representation Q of G, let $n_+(Q)$ be the number of times the Q representation appears in the harmonic expansion of positive chirality spinors, and let $n_-(Q)$ be the number of times the Q representation appears in the expansion of negative chirality spinors. By standard theorems about homogeneous spaces and elliptic operators, $n_+(Q)$ and $n_-(Q)$ are always finite, and $n_+(Q) = n_-(Q)$ for all but finitely many Q. Moreover, $\text{index}_Q(i\not{D}) = n_+(Q) - n_-(Q)$. After all, the index is the difference between the number of positive and negative chirality multiplets. This difference normally must be regularized since there are infinitely many states in Hilbert space, and the regularization usually involves pairing off the states of equal, nonzero energy. In a homogeneous space, working in a subspace of Hilbert space defined by a definite representation, Q reduces the problem to a finite-dimensional problem. No regularization is needed; $\text{index}_Q(i\not{D})$ is just the difference $n_+(Q) - n_-(Q)$ between the number of positive chirality and negative chirality Q multiplets.

Let us illustrate these ideas with some simple examples. Consider first a particle of spin $\frac{1}{2}$ moving on the ordinary 2-dimensional sphere S^2. Let **J** be the angular momentum operator, and define the "helicity" of a particle at **x** to be the component of angular momentum about the **x** axis. It equals $\pm\frac{1}{2}$; it is $+\frac{1}{2}$ for states of positive chirality, $-\frac{1}{2}$ for states of negative chirality.

It is well known that a particle of helicity h can be in a state of total angular momentum $J = |h|, |h| + 1, |h| + 2, \ldots$, with each allowed value of J appearing exactly once in the harmonic expansion. This is why $J = 1$ is the lowest possible value for photons, and $J = 2$ is the lowest possible value for gravitons.

Whether the chirality is positive or negative, the absolute value of the helicity of a spin $\frac{1}{2}$ particle is $\frac{1}{2}$. So the allowed values of angular momentum are the same for each chirality:

$$\begin{aligned} \hat{\Gamma} = +1: &\quad J = \tfrac{1}{2}, \tfrac{3}{2}, \tfrac{5}{2}, \ldots, \\ \hat{\Gamma} = -1: &\quad J = \tfrac{1}{2}, \tfrac{3}{2}, \tfrac{5}{2}, \ldots. \end{aligned} \tag{14}$$

Now we can see that the Dirac operator on the sphere has no stable spectrum of zero modes. Since the Dirac operator commutes with J but reverses chirality, acting on (say) the multiplet of given J and chirality ± 1, the Dirac operator gives either zero or else the multiplet of the same J and chirality ∓ 1 (see figure 3). Since the Dirac operator is hermitian, it either exchanges these two multiplets or annihilates both of them. Therefore the positive chirality zero modes have the same eigenvalues of J as the negative chirality zero modes, and the character-valued index vanishes.

This result could be obtained in various other ways. It follows from the fact that the rotation group $SU(2)$ has no complex representations, or from the fact that a reflection of the two-sphere reverses parity and exchanges the

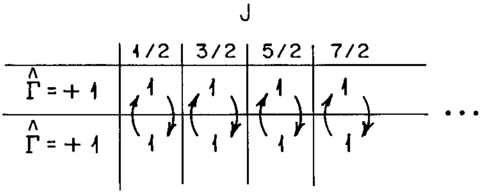

Figure 3
The angular momentum and chirality spectrum of the Dirac operator on a sphere. The Dirac operator permutes the states in the way indicated by the arrows.

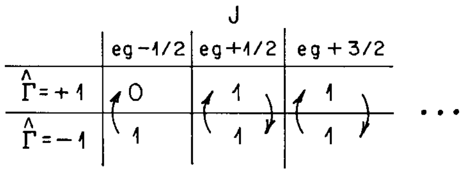

Figure 4
The quantum numbers of a charged Dirac particle on a sphere in the presence of a magnetic monopole field.

two chiralities, or from the fact that the two-sphere with its usual metric has positive scalar curvature so that (by Lichnerowicz's theorem) the Dirac operator has no zero modes at all. Now, however, we shall consider a slightly modified problem in which the character-valued index is nonzero.

Place at the center of the sphere a magnetic monopole of strength $eg = n/2$, for some integer n. The angular momentum operator now acquires an extra piece [25] $eg\hat{x}$ related to the quantization of magnetic charge. This adds eg to the fermion helicity, so that a fermion of chirality $+1$ has effective helicity $eg + \frac{1}{2}$ and a fermion of chirality -1 has effective helicity $eg - \frac{1}{2}$. If, say, $eg > 0$, the allowed values of angular momentum are now

$$\begin{aligned} \hat{\Gamma} = +1: &\qquad J = eg + \tfrac{1}{2}, eg + \tfrac{3}{2}, eg + \tfrac{5}{2}, \ldots, \\ \hat{\Gamma} = -1: &\qquad J = eg - \tfrac{1}{2}, eg + \tfrac{1}{2}, eg + \tfrac{3}{2}, eg + \tfrac{5}{2}, \ldots. \end{aligned} \tag{15}$$

The crucial point is now that states of $J = eg - \frac{1}{2}$ exist for chirality -1 but not for chirality $+1$. The Dirac operator must annihilate these states, because acting on states of $\hat{\Gamma} = -1$, $J = eg - \frac{1}{2}$, the Dirac operator would give states of $\hat{\Gamma} = +1$ and $J = eg - \frac{1}{2}$, and such states do not exist. Other multiplets cancel out as before (figure 4), so the stable spectrum of zero modes is a single multiplet of $\hat{\Gamma} = -1$ and $J = eg - \frac{1}{2}$.

We shall obtain this answer in a different way in section IV as an illustration of a much more general and powerful method of calculating the character-valued index of an operator. The technique for harmonic expansions for G-invariant operators on a homogeneous space G/H does not seem to be well known among physicists. It is explained, for instance, in appendix IV of Salam and Strathdee in [3]. In later sections we shall occasionally state without detailed derivation results obtainable from harmonic expansions.

IV The Atiyah-Hirzebruch Theorem

We now turn to the proof of the Atiyah-Hirzebruch theorem, which states that the character valued index of the Dirac operator vanishes on any manifold with a continuous symmetry group (in any even number of dimensions, though our main interest is $d = 4k + 2$). The presentation will parallel a recent treatment of Morse theory [16] and is essentially a more concrete version of the original proof.

Let B be a compact Riemannian manifold of even dimension n. Suppose B admits the action of a symmetry group G. We wish to prove that index $(g) = 0$ for every $g \in G$. Since every element of G can be approximated arbitrarily well by elements of suitably chosen $U(1)$ subgroups of G, it suffices to prove that index$(h) = 0$ whenever h is an element of any $U(1)$ subgroup R of G. We therefore specialize to the case of a $U(1)$ symmetry group R. Since the representations of $R \cong U(1)$ are labeled by an integer or half-integer n,[8] it suffices to show that index$_n(i\not{D}) = 0$ for all n.

Let ϕ^i be a local coordinate system of B. Let $K^i(\phi^j)$ be the Killing vector field that generates R. [This means that, infinitessimally, the R transformation is $\phi^i \to \phi^i + \varepsilon K^i(\phi^j)$.] Acting on spinors, the generator of R is the "Lie derivative" operator:

$$\mathscr{L}_K = i(K^i D_i + \tfrac{1}{4}\Gamma^{ij}(D_i K_j)), \tag{16}$$

where $\Gamma^{ij} = \frac{1}{2}[\Gamma^i, \Gamma^j]$. Using the Killing vector equation $D_i K_j + D_j K_i = 0$ and standard identities, it is not difficult to verify that $\mathscr{L}_K$ and $i\not{D}$ commute—as should be the case since $\mathscr{L}_K$ generates a symmetry. Therefore we may study simultaneous solutions of the equations

$$\mathscr{L}_K \psi = n\psi, \qquad i\not{D}\psi = \lambda\psi. \tag{17}$$

8. The eigenvalues on spinor states are half-integers in certain cases.

Our problem is to show that the Dirac index $\text{index}_n(i\not{D})$ vanishes for each sector of Hilbert space labeled by the integer (or half-integer) n.

The basic property of this index is that it is invariant under arbitrary deformations of the operator $i\not{D}$ that are $U(1)$ invariant and preserve the property $i\not{D}\hat{\Gamma} = -\hat{\Gamma} i\not{D}$. Let us therefore perturb the Dirac operator in a way that preserves these properties and simplifies the analysis of it spectrum. Instead of $i\not{D}$ we shall study

$$i\not{D}_t = i\not{D} + t\Gamma^i K_i, \tag{18}$$

where t is a conveniently chosen real number. The character-valued index of $i\not{D}_t$ must be independent of t. We shall prove that $\text{index}_n(i\not{D})$ vanishes for $n \geqslant 0$ by studying the behavior as $t \to +\infty$, and we shall prove that $\text{index}_n(i\not{D})$ vanishes for $n \leqslant 0$ by studying the behavior as $t \to -\infty$.

We define a "Hamiltonian"

$$H_t = (i\not{D}_t)^2 = (i\not{D})^2 + t^2K^2 + 2itK^jD_j + it\Gamma^{ij}D_iK_j. \tag{19}$$

If we could show that for sufficiently large t, H_t has no zero eigenvalues, this would establish the vanishing of the character-valued index. The general reason this might be true is that the t^2K^2 term is positive definite and becomes very large for large t. However, the analysis is made subtle by the term $2itK^jD_j$, which is not positive definite and can have large matrix elements.

A crucial observation is that in the sector $\mathscr{L}_K\psi = n\psi$, H_t reduces to

$$H_t^{(n)} = (i\not{D})^2 + t^2K^2 + 2tn + \tfrac{1}{2}it\Gamma^{ij}D_iK_j. \tag{20}$$

Since t is freely at our disposal, we choose $t > 0$ if $n \geqslant 0$ and $t < 0$ if $n < 0$. In this way the term $2tn$, as well as the t^2K^2 term, is positive.

Now, if the Killing vector field K^i has no zeros, then all eigenvalues are of order t^2 as $t \to \infty$. In this case, the character-valued index certainly vanishes. In general, however, K^i vanishes at certain points, and our analysis is more difficult.

For large $|t|$, the spectrum of $H_t^{(n)}$ can be calculated in an asymptotic expansion in powers of $1/|t|$ by expanding near the minima of the potential. The relevant minima (which might give states that do not diverge in energy as $|t| \to \infty$) are zeros of K. For simplicity, we shall treat the case of an isolated zero of K, but the general case is not much different.

We may take our isolated zero of K to be at $\phi^i = 0$. Near $\phi^i = 0$ we can choose the locally euclidean coordinates ϕ^i to be such that $K_i = \omega_{ij}\phi^j +$

$O(\phi^2)$, with ω_{ij} a constant matrix,

$$\omega_{ij} = \begin{bmatrix} 0 & r_1 & & & & & & \\ -r_1 & 0 & & & & & & \\ & & 0 & r_2 & & & & \\ & & -r_2 & 0 & & & & \\ & & & & \ddots & & & \\ & & & & & \ddots & & \\ & & & & & & 0 & r_{d/2} \\ & & & & & & -r_{d/2} & 0 \end{bmatrix}. \tag{21}$$

[Here d is the dimension of our manifold; the r_i are integers since K generates a $U(1)$ group.] For large $|t|$, by keeping only terms that contribute an amount of order t to the energy, $H_t^{(n)}$ simplifies to

$$\begin{aligned} H_t^{(n)} = & -\sum_{i=1}^{d} \frac{\partial^2}{\partial \phi_i{}^2} + \sum_{l=1}^{d/2} t^2 r_l^2 (\phi_{2l-1}^2 + \phi_{2l}^2) + 2tn \\ & - t \sum_{l=1}^{d/2} \frac{ir_l}{2} [\Gamma_{2l-1}, \Gamma_{2l}]. \end{aligned} \tag{22}$$

The ground state energy of (22) is easily calculated. The first two terms are a sum of commuting harmonic oscillator Hamiltonians. The matrices $(i/2)[\Gamma_{2l-1}, \Gamma_{2l}]$ commute with each other and with the rest of the Hamiltonian, and have eigenvalues ± 1. The ground state energy of (22) is $|t| \sum_{i=1}^{d/2} |r_i| + 2tn$.[9] This is a good approximation for large $|t|$. Considering $t \to +\infty$ if $n \geqslant 0$, and $t \to -\infty$ if $n < 0$, we see that $H_t^{(n)}$ has no zero eigenvalues and consequently that the character-valued index of $i\not{D}$ vanishes.

If the zeros of K are not isolated points, the discussion must be changed only slightly. Let F be any connected component of the submanifold on which K vanishes. The potential $t^2 K^2$ vanishes on F, so our spin $\frac{1}{2}$ particle moves freely along F, but the motion orthogonal to F is restricted to a distance of order $1/\sqrt{t}$. The orthogonal motion is governed by an operator similar to (20), and the zero point energy of the orthogonal motion is strictly

9. If we restrict ourselves to states of $\mathscr{L}_K \psi = n\psi$ (as we should), the ground state energy of (22) is even larger. This of course does not change the conclusion.

positive if $|t| \to \infty$ with $tn \geqslant 0$, again showing that the character-valued index vanishes. (The discussion of degenerate Morse theory in reference [16] is similar.) This completes the proof of the Atiyah-Hirzebruch theorem.

Let us, however, now look at the preceding formulas from a different viewpoint, the goal being to obtain the fixed point formula associated with the Atiyah-Singer index theorem [26]. As we shall see, this formula is a powerful tool for computing the character valued index when it is not zero.

Let us now study H_t for $t \to +\infty$. Low-lying eigenvalues of H_t are concentrated near zeros of K, which for simplicity we take to be isolated points. Most of these states have energy of order t, but some have energy that vanishes as $t \to \infty$. (This has been obscured in the presentation until now.) Let $a^{(i)}_{n,\pm}$ be the number of state ψ concentrated near the ith zero whose energy does not diverge as $t \to +\infty$ and that obey $\bar{\Gamma}\psi = \pm\psi$ and $\mathscr{L}_K\psi = n\psi$. (We have proved $a^{(i)}_{n,\pm} = 0$ for $tn \geqslant 0$.) Define the "local index" of the Dirac operator at the ith zero as

$$f_i(\theta) = \sum_n e^{in\theta}(a^{(i)}_{n,+} - a^{(i)}_{n,-}). \tag{23}$$

The character-valued index $I(\theta)$ is obtained as the sum of the local indexes,

$$I(\theta) = \sum_i f_i(\theta), \tag{24}$$

since this sum includes the contribution of all states whose energy is not of order $|t|$, and only such states can contribute to $I(\theta)$. Of course, we have proved that the Dirac case has $I(\theta) = 0$, so (24) is a set of restrictions on the $a^{(i)}_{n,\pm}$. However, we shall obtain a formula like (24) for other problems in which $I(\theta) \neq 0$.

To compute the $a^{(i)}_{n,\pm}$, we could simply diagonalize the harmonic oscillator Hamiltonian (22). A more efficient method is as follows. Zero energy states must obey $i\not{D}_t\psi = 0$. Here $i\not{D}_t = i\Gamma^j(D_j - itK_j)$ has a very simple interpretation; it corresponds to a particle interacting with the abelian vector potential $A_j = tK_j$ as well as the metric of the curved manifold under study. For large t, we know the low-lying states are concentrated near zeros of K. If near a zero at (say) $\phi = 0$, $K^i = \omega^{ij}\phi_j$, with ω^{ij} a constant matrix, then the "magnetic field" $F_{ij} = \partial_j A_j - \partial_j A_i$ is just the constant matrix $t\omega_{ij}$. So as $t \to \infty$, our problem reduces to the study of the Dirac equation in a constant magnetic field. As in the usual 3-dimensional case, the ground state energy is zero (but states of zero energy have $n < 0$ if $t \to +\infty$ or $n > 0$ if $t \to -\infty$).

With ω_{ij} in the canonical form (21), the analysis is very simple. Define

$$
\begin{aligned}
i\not{D}_t^{(1)} &= i \sum_{i=1,2} \Gamma^i(D_i + itK_i),\\
iD_t^{(2)} &= i \sum_{i=3,4} \Gamma^i(D_i + itK_i),\\
&\vdots\\
iD_t^{(n/2)} &= i \sum_{i=n-1,n} \Gamma^i(D_i + itK_i).
\end{aligned}
\tag{25}
$$

Then $(i\not{D}_t)^2 = \sum_{j=1}^{n/2} (i\not{D}_t^{(j)})^2$, so a solution of $i\not{D}_t\psi = 0$ obeys simultaneously $i\not{D}_t^{(1)}\psi = \cdots = i\not{D}_t^{(n/2)}\psi = 0$. Thus, we need only the well known solution of the problem of a constant magnetic field in *two* dimensions.

Choose a basis of gamma matrices

$$
\Gamma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \Gamma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.
$$

If the "rotation angle" is r, so $K_1 = r\phi_2, K_2 = -r\phi_1$, then the 2-dimensional Dirac operator is

$$
\not{D}_t = \begin{pmatrix} 0 & (\partial_1 - \operatorname{tr}\phi_1) - i(\partial_2 - \operatorname{tr}\phi_2) \\ (\partial_1 + \operatorname{tr}\phi_1) + i(\partial_2 + \operatorname{tr}\phi_2) & 0 \end{pmatrix}. \tag{26}
$$

With $\psi = \binom{u}{v}$, $\not{D}_t\psi = 0$ if

$$
\begin{aligned}
u(\phi_1, \phi_2) &= (\phi_1 + i\phi_2)^k \exp -\operatorname{tr}(\phi_1^2 + \phi_2^2),\\
v(\phi_1, \phi_2) &= (\phi_1 - i\phi_2)^k \exp +\operatorname{tr}(\phi_1^2 + \phi_2^2),
\end{aligned}
\tag{27}
$$

where $k = 0, 1, 2, \ldots$. The chirality operator is

$$
\hat{\Gamma} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
$$

For $t \to +\infty$, if $\operatorname{tr} > 0$, only positive chirality zero modes are normalizable. If $\operatorname{tr} < 0$, only negative chirality zero modes are normalizable.

The symmetry generator is

$$
\begin{aligned}
\mathscr{L}_K &= i(\Gamma^i D_i + \tfrac{1}{4}\Gamma^{ij} D_i K_j)\\
&= -ir(\phi_1\,\partial_2 - \phi_2\,\partial_1) + \frac{r}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\end{aligned}
\tag{28}
$$

We see that on the states (27), $\mathscr{L}_K = r\hat{\Gamma}(k + \frac{1}{2})$.

The local index in the 2-dimensional problem would be $h(\theta) = \sum_n e^{in\theta}(a_{n,+} - a_{n,-})$, where $a_{n,\pm}$ are the number of zero energy states ψ with $\mathscr{L}_K\psi = n\psi$, $\hat{\Gamma}\psi = \pm\psi$. Using (27) and (28), we see that if $r > 0$,

$$\begin{aligned} h(\theta) &= e^{i(r\theta/2)} + e^{i(3r\theta/2)} + e^{i(5r\theta/2)} + \cdots \\ &= \frac{e^{ir\theta/2}}{1 - e^{ir\theta}} \\ &= \frac{i}{2}\frac{1}{\sin\frac{1}{2}r\theta}. \end{aligned} \tag{29}$$

If $r < 0$, we get

$$\begin{aligned} h(\theta) &= -e^{-ir\theta/2} - e^{-i(3r\theta/2)} - e^{-i(5r\theta/2)} - \cdots \\ &= -\frac{e^{-ir\theta/2}}{1 - e^{-ir\theta}} = \frac{i}{2}\frac{1}{\sin\frac{1}{2}r\theta}. \end{aligned} \tag{30}$$

It is one of the wonders of analytic functions that these expressions are equal, so we need not worry about the sign of r.

In view of the separation of variables (zero eigenvalues of $\not{D}_t$ are zero eigenvalues of each of the $\not{D}_t^{(n)}$), we can now easily compute the local index $f_i(\theta)$ at the ith zero. It is just

$$f_i(\theta) = \prod_{a=1}^{n/2}\left(\frac{i}{2}\frac{1}{\sin\frac{1}{2}r_a^{(i)}\theta}\right), \tag{31}$$

where $r_a^{(i)}$ is the ath rotation angle at the ith zero of k. The character-valued index of the Dirac operator is therefore

$$I(\theta) = \left(\frac{i}{2}\right)^{n/2}\sum_i\prod_a\frac{1}{\sin\frac{1}{2}r_a^{(i)}\theta}, \tag{32}$$

which is the fixed point formula. Although we obtained it by taking $t \to +\infty$, an analysis for $t \to -\infty$ leads to the same formula.

Since we know that $I(\theta) = 0$, (32) is a set of very restrictive conditions on the $r_a^{(i)}$. Actually the Atiyah-Hirzebruch theorem is an easy consequence of (32). The right-hand side of (32) defines a rational function of $w = e^{i\theta/2}$ that vanishes at $w = 0$ and $w = \infty$. This function has no poles. [Individual terms in (32) have poles at $|w| = 1$. These poles must cancel after summing over fixed points, for the following reason. An elliptic operator always has only finitely many zero modes, so $I(\theta)$ has an expansion $I(\theta) = \sum_n a_n e^{in\theta}$ with only finitely many nonzero a_n; therefore $I(\theta)$ is always nonsingular for real θ]. A rational function without poles that vanishes at infinity is zero, so $I(\theta) = 0$.

For our purposes the virtue of (32) is that it generalizes easily to other problems. Suppose we wish to study a field $\psi_{\alpha A}$ with a spinor index α and some other index A. For instance, A may be a vector index if we wish to study the spin $\frac{3}{2}$ field; or A may be a Yang-Mills index. We wish to calculate the character-valued index of an operator acting on $\psi_{\alpha A}$. The physically relevant operator may not actually be the Dirac operator $i\Gamma^j D_j$, but we shall assume it differs from the Dirac operator only by irrelevant nonminimal terms.

As before, the character-valued index may be computed from the large $|t|$ limit of $i\not{D}_t = i\Gamma^j(D_j + itK_j)$. The index A only enters in the connection used to define D_j, but the connection is irrelevant as $|t| \to \infty$, as shown by the reduction to a flat space harmonic oscillator problem. So our previous determination of the spectrum is still valid.

What is different is the determination of the quantum numbers of the low-lying states. The symmetry generator is now

$$\tilde{\mathscr{L}}_K = \mathscr{L}_K + Q(\phi^k) \qquad (33)$$

with an extra term Q (without derivatives) that acts on the A index. (It is a generalization of the extra angular momentum term for a charge interacting with a magnetic monopole.) Let $Q^{(i)}$ be the value of Q at the ith zero of K. The states near the ith zero that have approximately zero energy are still given by (27), but near the ith zero, $\tilde{\mathscr{L}}_k$ is $r\hat{\Gamma}(k + \frac{1}{2}) + Q^{(i)}$. The effect is very simple. In the sums (29) and (30), one has an extra factor $\mathrm{Tr}\, e^{i\theta Q^{(i)}}$ (the trace being over the A index), so now

$$f_i(\theta) = \left(\frac{i}{2}\right)^{n/2} \mathrm{Tr}\, e^{i\theta Q^{(i)}} \prod_{a=1}^{n/2} \frac{1}{\sin \frac{1}{2} r_a^{(i)} \theta} \qquad (34)$$

and the fixed point formula is

$$I(\theta) = \left(\frac{i}{2}\right)^{n/2} \sum_i \mathrm{Tr}\, e^{i\theta Q^{(i)}} \prod_{a=1}^{n/2} \frac{1}{\sin \frac{1}{2} r_a^{(i)} \theta}. \qquad (35)$$

We shall not write down here the more general formula that holds if the zeros of K are not isolated. This formula involves weighting the factors of

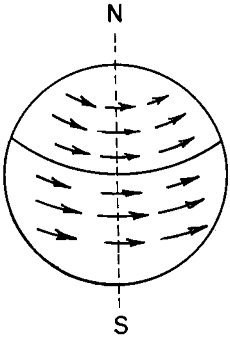

Figure 5
K is taken to generate the rotation of a sphere.

$1/(\sin\frac{1}{2}r\theta)$ by the number of zero modes of a certain Dirac operator on the fixed point set.

One may wonder "why" a fixed point formula exists. The character-valued index is formally $\mathrm{Tr}\,\hat{\Gamma}e^{i\theta\mathscr{L}_K}$ (since states of nonzero energy are paired and cancel out of the trace). The trace of a matrix is the sum of the diagonal matrix elements. In the coordinate basis, the diagonal matrix elements of $e^{i\theta\mathscr{L}_K}$ vanish except near the zeros of K. The fixed point formula is similar to the method of Landau and Lifshitz[27] for computing the character of a molecular symmetry group furnished by the molecular vibrations in terms of fixed points of the symmetry group action.

To gain some practice with (35), let us use these methods to retrieve the results of section III. We consider a spin $\frac{1}{2}$ particle moving on the two-sphere. We take K to be (figure 5) the generator of a rotation about the z axis. There are two fixed points, the north pole N and the south pole S. A rotation that is counterclockwise as seen by an observer looking down at N is clockwise to an observer looking down at S. So the rotation angles are $r = 1$ at N and $r = -1$ at S. The fixed point formula gives

$$I(\theta) = \frac{i}{2}\frac{1}{\sin\frac{1}{2}\theta} + \frac{i}{2}\frac{1}{\sin(\frac{1}{2}\theta)} = 0, \tag{36}$$

as expected. Now, as in section III, we assume our spin $\frac{1}{2}$ particle to be charged, and we place at the center of the sphere a magnetic monopole with $eg = n/2, n \in Z$. It is well known[25] that the angular momentum operator is shifted: $\mathbf{J} \to \mathbf{J} + eg\mathbf{x}$. In our case $\mathscr{L}_k$ is J_z; the operator Q is the extra piece in

J_z or egz. At N, $z = 1$; at S, $z = -1$. Note that the $Q^{(i)}$ are numbers, $\pm eg$, not matrices, since in the $U(1)$ case the charge index A has only one value. The fixed point formula is now

$$\begin{aligned} I(\theta) &= \frac{i}{2} e^{ieg\theta} \frac{1}{\sin(\frac{1}{2}\theta)} + \frac{i}{2} e^{-ieg\theta} \frac{1}{\sin(\frac{1}{2}\theta)} \\ &= -\left(e^{ieg\theta} - e^{-ieg\theta}\right) \frac{e^{-i\theta/2}}{1 - e^{-i\theta}} \\ &= - \sum_{n=-[eg-(1/2)]}^{eg-(1/2)} e^{in\theta}. \end{aligned} \tag{37}$$

This agrees with our result from section III, since the sum in (37) is the trace of $e^{i\theta J_z}$ in the represention of $J = eg - \frac{1}{2}$.

The proof of the Atiyah-Hirzebruch theorem that we have given is closely related to the original argument. It has the virtue of yielding the fixed point formula, which has many other applications, as we shall see. If one is only interested in the vanishing of the character-valued index of the Dirac operator, the following alternative argument may be sketched. For a space B that admits action of a nonabelian group G, the Lawson-Yau theorem[22] states that B admits a G-invariant metric of positive scalar curvature. (The basic idea of the proof is as follows. *Any* G-invariant metric g can be decomposed as $g = g_1 + g_2$, where g_1 is the metric transverse to the directions of the group action and g_2 is the metric along the group action. Lawson and Yau show that the metric $g^\varepsilon = g_1 + \varepsilon g_2$ has positive scalar curvature if ε is suitably small and positive and g_1 obeys some mild conditions at the fixed points of G.) This, combined with the Lichnerowicz theorem (no zero modes of the Dirac operator if the scalar curvature is positive), implies the vanishing of the character-valued Dirac index for manifolds with nonabelian symmetry groups. For manifolds with only an abelian symmetry group, one may reason as follows. Let B be a manifold with $U(1)$ symmetry that violates the Atiyah-Hirzebruch theorem in n dimensions. Let $\tilde{B}$ be a manifold of dimension $n + 2$ defined to be a nontrivial fiber bundle over S^2 with fiber B. [To construct this bundle, consider the space S^3 of pairs of complex numbers $\binom{z_1}{z_2}$ with $|z_1|^2 + |z_2|^2 = 1$. In the product $S^3 \times B$, make the identification

$$\left(\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \phi^i\right) \simeq \left(\begin{pmatrix} z_1 e^{i\alpha} \\ z_2 e^{i\alpha} \end{pmatrix}, e^{in\alpha \mathscr{L}_K} \phi^i\right)$$

for suitable n.] The space $\tilde{B}$ has $SU(2) \times U(1)$ symmetry, and has a nontrivial character-valued index of the Dirac operator if B does. (To see this, choose on $\tilde{B}$ a Kaluza-Klein metric with large radius in the S^2 directions and tiny radius in the B directions and solve the Dirac equation on $\tilde{B}$ in a Born-Oppenheimer–Kaluza-Klein approximation.) Since the Lawson-Yau and Lichnerowicz theorems imply that the character-valued index must vanish on $\tilde{B}$, it must also vanish on B.

V Rarita-Schwinger Fields

In this section we shall study the character-valued index of the Rarita-Schwinger field. We shall not be able to reach a comprehensive result similar to the Atiyah-Hirzebruch theorem for the Dirac case. We shall prove that the zero modes of the Rarita-Schwinger operator on any homogeneous space G/H form a real representation of G. In other words, we shall show that on any homogeneous space of dimension $4k + 2$, the character-valued index vanishes. For homogeneous spaces of dimension $4k$, the same argument shows that the character-valued index vanishes except for the trivial character (the topological index). Unfortunately, I do not know a general result for the Rarita-Schwinger field on spaces that are not homogeneous spaces. (I also have been unable, despite many attempts, to find a case in which the character-valued index is nontrivial, and I believe that if such manifolds exist, they are rather complicated manifolds with rather small symmetry groups in relation to the number of dimensions.) Because our results will not be entirely conclusive, we shall return after discussing the theorem on homogeneous spaces to a discussion of various problems in the use of the Rarita-Schwinger operator.

Our basic tool will be the fixed point formula discussed in the previous section. In particular, we shall not use the local supersymmetry of the Rarita-Schwinger field; it may be possible to find a stronger result by using this property.

In an appropriate gauge, the Rarita-Schwinger field is simply a vector spinor field $\psi_{\mu\alpha}$ (μ is a vector index, α a spinor index) that obeys, up to irrelevant nonminimal terms, a Dirac equation $\not{D}\psi_\mu = 0$. Of course, we wish to discard zero modes of ψ_μ that can be gauged away or that violate gauge conditions. Physically, in quantizing a theory, zero modes that are gauge artifacts are canceled by zero modes of the spin $\frac{1}{2}$ ghost fields. Therefore, we must subtract from the character-valued index of the Rarita-Schwinger field the corresponding index of the spin $\frac{1}{2}$ ghosts. (This is the general logic, but actually the ghost index vanishes, by the Atiyah-Hirzebruch theorem.)

Let us work out the fixed point formula for the spin $\frac{3}{2}$ field. Consider an isolated zero of the Killing vector field K. Suppose that near the zero (which we assume to be at $\phi^i = 0$), $K_i = \omega_{ij}\phi^j$, where ω is a constant antisymmetric matrix,

$$\omega = \begin{pmatrix} 0 & r_1 & & & & & & \\ -r_1 & 0 & & & & & & \\ & & 0 & r_2 & & & & \\ & & -r_2 & 0 & & & & \\ & & & & \ddots & & & \\ & & & & & & 0 & r_{n/2} \\ & & & & & & -r_{n/2} & 0 \end{pmatrix}. \tag{38}$$

The symmetry generator $\tilde{\mathscr{L}}_K$ for a spin $\frac{3}{2}$ field is

$$\begin{aligned} \tilde{\mathscr{L}}_L\psi_j &= i(K^i D_i\psi_j + \tfrac{1}{4}\Gamma^{lm}(D_l K_m)\psi_j) - i(D_j K^i)\psi^i \\ &= \mathscr{L}_K\psi_j - i(D_j K^i)\psi^i, \end{aligned} \tag{39}$$

where $\mathscr{L}_K$ is the generator for a spin $\frac{1}{2}$ field and the extra piece acting on the vector index is $-i(D_j K^i)$. At $\phi^i = 0$, $D_j K_i = \omega_{ij}$, so $\tilde{\mathscr{L}}_K$ acts on the vector index of ψ_i by multiplication by the matrix $i\omega$. The factor of $\mathrm{Tr}\, e^{i\theta Q^{(n)}}$ in formula (34) for the local index is here to be replaced by

$$\mathrm{Tr}\, e^{i\theta(i\omega)} = 2\sum_{a=1}^{n/2} \cos\theta r_a. \tag{40}$$

The fixed point formula for the character-valued index of the Rarita-Schwinger field is then

$$I(\theta) = \left(\frac{i}{2}\right)^{n/2} \sum_i \left[\left(\sum_{a=1}^{n/2} \frac{1}{\sin\frac{1}{2}r_a^{(i)}\theta}\right)\left(\sum_{b=1}^{n/2} 2\cos r_b^{(i)}\theta - 1\right)\right]. \tag{41}$$

Here i runs over the fixed points or zeros of K; $r_a^{(i)}$, $a = 1, \ldots, n/2$, are the rotation angles at the ith zero; and the minus one in the last factor in (41) is chosen to subtract the index of the ghost fields, as discussed earlier. (Minus one equals minus two plus one; for the quantization of ψ_i, there are[6] two ghosts with the same chirality as ψ_i and one of opposite chirality.) Equation (41) has a generalization when the fixed points are not isolated.

Equation (41) places very severe restrictions on the possibility of obtaining a complex representation of Rarita-Schwinger zero modes; it shows that $I(\theta)$ is a rational function of $w = e^{i\theta/2}$ with poles only for $|w| = 1$ or at $w = 0$ or $w = \infty$. The poles at $|w| = 1$ must cancel upon summing over fixed points, as in the spin $\frac{1}{2}$ case. [Note the discussion following (32).] The rational function $I(w)$ must be a constant unless there really are poles at $w = 0$ or $w = \infty$. If the largest rotation angle at the ith fixed point is $r_b^{(i)}$, then the contribution of this fixed point to $I(w)$ behaves for $w \to \infty$ or $w \to 0$ as

$$f_i(w) \sim w^{(r_b^{(i)} - \sum_{a \neq b} r_a^{(i)})}. \tag{42}$$

There are poles at $w = 0$ and $w = \infty$ if and only if the largest rotation angle $r_b^{(i)}$ is bigger than the sum of the others:

$$r_b^{(i)} > \sum_{a \neq b} r_a^{(i)}. \tag{43}$$

If (43) is not obeyed for any i, $I(w)$ has no singularities. Even if (43) is obeyed, the poles may vanish in summing over i. It is difficult to satisfy (43) in a multidimensional space with many rotation angles at each fixed point.

If for each i and each b

$$r_b^{(i)} \leqslant \sum_{a \neq b} r_a^{(i)}, \tag{44}$$

then $I(w)$ is a rational function without poles and bounded for $w \to \infty$, and therefore is a constant. The constant is an integer—the ordinary or topological index of the Rarita-Schwinger field. If (44) is always a strict inequality, $r_b^{(i)} < \sum_{a \neq b} r_a^{(i)}$ for all i and b, then $I(w)$ vanishes as $w \to \infty$, so $I(w) = 0$ and the topological index vanishes.

We shall use these considerations to prove that the character-valued index of the Rarita-Schwinger field on a homogeneous space always vanishes except possibly for the trivial character (which may appear in the case of $4k$ dimensions).

The homogeneous space G/H is defined as follows. It is the space of all $g \in G$ with g and gh considered equivalent for any $h \in H$. Because of the equivalence relation, the dimension of G/H equals the dimension of G minus the dimension of H. The space G/H is invariant under $g \to ug$ for any $u \in G$. A fixed point of this transformation is an element g of G such that $ug = gh$ for some $h \in H$; in other words $g^{-1}ug \in H$. If u is not equivalent up to similarity to an element of H, then the symmetry transformation $g \to ug$ has no fixed points.

If the rank of H is less than the rank of G, then the generical generator A

of the Cartan subalgebra of G is not equivalent (up to similarity) to any generator of H. Then $e^{i\theta A}$ acts on G/H without fixed points, so the character-valued index vanishes. Hence we need only consider the case rank H = rank G.

Let us now make a brief detour. In general, given two spaces M, N the character-valued Dirac and Rarita-Schwinger (R.S.) indexes of M, N, and the product $M \times N$ are related by

$$\begin{aligned} \text{index}_{\text{Dirac}}(M \times N) &= \text{index}_{\text{Dirac}}(M) \cdot \text{index}_{\text{Dirac}}(N), \\ \text{index}_{\text{R.S.}}(M \times N) &= \text{index}_{\text{R.S.}}(M) \cdot \text{index}_{\text{Dirac}}(N) \\ &\quad + \text{index}_{\text{Dirac}}(M) \cdot \text{index}_{\text{R.S.}}(N) \\ &\quad + \text{index}_{\text{Dirac}}(M) \cdot \text{index}_{\text{Dirac}}(N). \end{aligned} \tag{45}$$

These equations hold because the Dirac and Rarita-Schwinger equations on $M \times N$ can be solved by separation of variables. The second equation in (45) (which is our real interest) arises because the vector index of ψ_i must be tangent to either M or N, so a Rarita-Schwinger solution on $M \times N$ obeys the Dirac equation on M and the Rarita-Schwinger equation on N or vice versa. [The last term in the second equation in (45), which is not intuitively obvious, arises in subtracting the ghost contributions from the Rarita-Schwinger indexes.] In particular, (45) implies that if M and N have vanishing Dirac index, then $M \times N$ has vanishing Rarita-Schwinger index. Since a homogeneous space has vanishing Dirac index (by the Lichnerowicz or Atiyah-Hirzebruch theorems), a product of two homogeneous spaces has vanishing Rarita-Schwinger index.

From these facts it follows that the Rarita-Schwinger index of G/H vanishes unless G is simple. For suppose $G = G_1 \times G_2$ with nontrivial $G_1 \times G_2$. A subgroup of G of maximal rank is then necessarily $H = H_1 \times H_2$, where H_1 and H_2 are maximal rank subgroups of G_1 and G_2, respectively. Then $G/H = (G_1/H_1) \times (G_2/H_2)$ is a product of homogenous spaces, and has vanishing character-valued Rarita-Schwinger index.

We still must study G/H with G simple and H a subgroup of maximal rank. Let us first calculate the rotation angles at the fixed points for a typical infinitessimal transformation $g \to (1 + i\varepsilon A)g$, A being a generator of G. Since H is maximal, A is equivalent (by conjugation) to a generator of H, so we may assume A is actually such a generator. A typical fixed point is then $g = 1$; because of the equivalence under right multiplication by an element of h, the transformation $g \to (1 + i\varepsilon A)g$ is equivalent to

$$g \to (1 + i\varepsilon A)g(1 - i\varepsilon A). \tag{46}$$

This again makes it clear that $g = 1$ is a fixed point. The fixed point is isolated if A is chosen generically. The rotation angles at $g = 1$ may be computed as follows. The Lie algebra $\mathscr{G}$ of G can be decomposed as the Lie algebra $\mathscr{H}$ of H plus an orthogonal complement $\mathscr{K}:\mathscr{G} = \mathscr{H} \oplus \mathscr{K}$. Near $g = 1$, the generic element of G/H is $1 + ik$, with $k \in \mathscr{K}$. The transformation (46) acts on k by $k \to k + i\varepsilon\,[A, k]$. The rotation angles at $g = 1$ are therefore just the eigenvalues of A acting on $\mathscr{K}$ by conjugation.

The other fixed points are at points g_i such that $g_i^{-1} A g_i \in \mathscr{H}$. The rotation angles are the eigenvalues of $g_i^{-1} A g_i$ acting by conjugation on $\mathscr{K}$.

Now we are ready to prove that the Rarita-Schwinger index vanishes on homogeneous spaces, except for the trivial character. Before stating the argument in a general way let us first consider the case that G is $SU(N)$. Let Y_0 be the $SU(N)$ generator

$$Y_0 = \begin{pmatrix} \frac{1}{2} & & & & & & \\ & -\frac{1}{2} & & & & & \\ & & 0 & & & & \\ & & & 0 & & & \\ & & & & \cdot & & \\ & & & & & \cdot & \\ & & & & & & 0 \end{pmatrix}. \tag{47}$$

It generates an $SU(2)$ subgroup of $SU(N)$, so its eigenvalues in any representation are integers or half-integers. Let $Y = Y_0 + \varepsilon Q$, where Q is a generic $SU(N)$ generator that commutes with Y_0 and ε is a suitably small real number.

Consider the Rarita-Schwinger equation on some space homogeneous under an $SU(N)$ action. Let Λ^+ and Λ^- be the $SU(N)$ representations of positive and negative chirality zero modes. Define $I(\theta) = \mathrm{Tr}_{\Lambda+}\, e^{i\theta Y} - \mathrm{Tr}_{\Lambda-}\, e^{i\theta Y}$. In any nontrivial $SU(N)$ representation the biggest eigenvalue of Y_0 is at least $\frac{1}{2}$. Hence, if Λ_+ differs from Λ_- by a nontrivial $SU(N)$ representation, the most positive power of $e^{i\theta}$ appearing will be at least $e^{i\theta\alpha}$, where $\alpha = \frac{1}{2} + O(\varepsilon)$. (For suitable Q and sufficiently small ε, there is no accidental cancellation of the highest power. This is the only role of Q and ε in the discussion.) This means that with $w = e^{i\theta/2}$, $I(w)$—if not a constant—diverges for $w \to \infty$ at least as $w^{1+O(\varepsilon)}$. Reference to (42) shows the necessary condition to achieve this; we need at one of the fixed points of the fixed points P of the transformation generated by Y

$$r_b - \sum_{a \neq b} r_a \geqslant 1, \tag{48}$$

where r_b is the largest rotation angle at P and r_a are the other rotation angles at P.

What are the rotation angles of the transformation Y that are not of order ε? They are one, corresponding to the $SU(N)$ generator

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & . & . & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & . & . & . & . & 0 \end{pmatrix} \tag{49}$$

and its adjoint, and $\frac{1}{2}$ corresponding to the generators

$$X_\alpha = \begin{pmatrix} 0 & 0 & a_1 & \dots & a_k \\ 0 & 0 & b_1 & \dots & b_k \\ 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix} \tag{50}$$

and their adjoints. Of course, in general we do not count A and all the X_α; we only count those that are in $\mathscr{K}$ (the complement in G of the H Lie algebra). The only way to obey (48) is to include A but none of the X_α. This means H must be a subgroup of $SU(N)$ that includes all the X_α and their adjoints and all the diagonal generators (since H has maximal rank) but not A. There is no such subgroup of $SU(N)$. Hence Λ_+ and Λ_- differ at most only by the trivial character.

The same argument goes through with $SU(N)$ replaced by any simple Lie group G. One simply replaces Y_0 by the generator of the Cartan subalgebra parallel to a root E of maximum length. E plays the role of A; the roots not orthogonal to E play the role of the X_α. The rest of the argument is unchanged.

Thus, we have shown that on any homogeneous space G/H (of dimension $4k$ or $4k + 2$) the character-valued index of the Rarita-Schwinger field is a constant, a multiple of the trivial character. It is equal simply to the ordinary index, the difference between the total number of right-handed and left-handed zero modes. The ordinary Rarita-Schwinger index certainly vanishes in $4k+2$ dimensions, but (unlike the ordinary Dirac index) it need *not* vanish on homogeneous spaces of dimension $4k$. It equals one on $SU(4)/(SU(2) \times SU(2) \times U(1))$, HP^2, and $G_2/O(4)$. This may be seen by the methods of sections III and IV, or from facts in reference [28], where properties of these spaces are described.

Interestingly, for spin greater than $\frac{3}{2}$, it *is* possible to obtain complex representations of zero modes on homogeneous spaces. Consider a spin $\frac{5}{2}$ field, which we may represent as a tensor spinor $\psi_{ij\alpha}$, i and j being vector indices and α a spinor index (we suppose $\psi_{ij} = \psi_{ji}$, $\psi i_i = 0$). The wave equation $\not{D}\psi_{ij} = 0$ *can* have zero modes in complex representations on homogeneous spaces. For instance, on the 6-dimensional manifold CP^3 this operator has the following stable spectrum of zero modes: one multiplet of left-handed zero modes in the symmetric tensor representation S^{ij}, and one multiplet of right-handed zero modes in the complex conjugate representation S^*_{ij}. This may be seen with the methods of sections III and IV.

Of course, physically sensible couplings of a massless spin $\frac{5}{2}$ field to gravity do not seem to exist[23]. However, this arises only because the timelike components ψ_{0i} of the tensor-spinor field have the wrong metric; because of the apparent nonexistence of spin $\frac{5}{2}$ gauge invariance, there is no way to cancel or remove them. Purely as a euclidean equation, with i and j tangent to the positive signature Kaluza-Klein space, the equation $\not{D}\psi_{ij} = 0$ makes perfect sense and has the properties just stated.

Our result about the Rarita-Schwinger operator is much less sweeping than the Atiyah-Hirzebruch theorem of the spin $\frac{1}{2}$ case. I do not know whether on some spaces that are not homogeneous the Rarita-Schwinger operator may have zero modes in complex representations. [I am convinced, from many unsuccessful attempts to find them, that if such spaces exist they are rather complicated. It is difficult to satisfy (42).]

There is actually a strategy that might very plausibly lead eventually to a general proof that the character-valued index of the Rarita-Schwinger field always vanishes except for the trivial character. The topological index of an operator is a cobordism invariant; this means that it vanishes for any manifold M of dimension n that is the boundary of a manifold of dimension $n+1$. The character-valued index is likewise invariant under equivariant cobordism; this means that if M admits the action of a group G and is the boundary of a manifold of dimension $n+1$ to which the G action on M can be extended, then the character-valued index vanishes for any operator on M. If a set of generators of the $U(1)$ spin bordism ring [the ring of spin manifolds with $U(1)$ symmetry modulo those which are boundaries] were found, our conjecture about the Rarita-Schwinger field could be proved by showing it to hold for all the generators. The mathematical problem of determining a set of generators for the (oriented) $U(1)$ spin bordism ring has not been solved. However, the analogous problem has been solved for the unoriented[44] and unitary[45] $U(1)$ bordism rings. These rings are generated by very simply spaces (essentially, homogeneous spaces and fiber bundles in which the fiber is a homogeneous space). If the $U(1)$ spin bordism ring is found to be generated by equally simple spaces, it will be possible to use the methods described previously to prove (or disprove) the conjecture that the character-valued Rarita-Schwinger index is always a constant.

Because the situation for spin $\frac{3}{2}$ fields is not completely clear, some general remarks on the subject may be useful. It is believed that massless spin $\frac{3}{2}$ fields can be consistently coupled to gravity only in locally supersymmetric theories. This apparently means[17] that we are limited to 11 dimensions or less. In addition, beyond 10 dimensions the chiral Rarita-Schwinger field has one loop anomalies[18] that spoil general covariance and cannot be canceled by the anomalies of any known fields that can be consistently coupled to gravity. For both of these reasons, it appears that 6 and 10 dimensions are the relevant cases for chiral Rarita-Schwinger fields. This corresponds to 2 or 6 compact dimensions, respectively.

With 2 compact dimensions, the only manifolds with continuous symmetry are the sphere, torus, and Klein bottle; the first two are homogeneous spaces, and on the last two the continuous symmetries have no fixed points, so on all of them the character-valued Rarita-Schwinger index vanishes. We turn then to the case of 6 compact dimensions.

Six compact dimensions are unfortunately too few to admit $SU(3) \times SU(2) \times U(1)$ symmetry[4]. One may be willing to postulate an elementary $U(1)$ gauge field and to try to obtain only $SU(3) \times SU(2)$ as the symmetry group of a six-manifold.[10] The unique six-manifold with $SU(3) \times SU(2)$ symmetry is $CP^2 \times S^2$. This is a homogeneous space to which our theorem applies; the character-valued index could not be nonzero. Even worse, this space does not admit spinors, so the Rarita-Schwinger equation on $CP^2 \times S^2$ cannot be defined[29].

However, if the $U(1)$ gauge field has a magnetic monopole expectation value on CP^2, spinors can be introduced (this is the so-called spin$_c$ structure); all fermi fields, including the Rarita-Schwinger field, must have

10. This suggestion was made independently by M. Gell-Mann.

nonzero (half-integral) $U(1)$ charges. The nonzero $U(1)$ charge of the spin $\frac{3}{2}$ field introduces new anomalies (the mixed gauge-gravity anomalies of reference [18]), which cannot cancel among themselves unless there are many more than two Weyl gravitinos (two is the maximum of any known or conjectured 10-dimensional supergravity theory) and which cannot be canceled by anomalies of spin $\frac{1}{2}$ fields (because of a different tensor structure). If we ignore this and proceed, we can calculate the spectrum of the Rarita-Schwinger operator on $CP^2 \times S^2$. The $SU(3) \times SU(2)$ invariant expectation value of the $U(1)$ field strength on $CP^2 \times S^2$ depends on two "monopole numbers"—a half-integer p on CP^2 (half-integer so as to get a spin$_c$ structure) and an arbitrary integer q on S^2. The resulting zero mode spectrum can be computed by the methods of sections III and IV. One obtains nontrivial complex representations, depending on p and q, but these representations have little resemblance to physics and are anomaly ridden [because the 10-dimensional theory with $U(1)$ coupling to the Rarita-Schwinger field is anomalous].

These accumulated difficulties may encourage us to give up on accommodating $SU(3) \times SU(2)$ symmetry in 6 compact dimensions. We may simply try for $SU(3)$ symmetry. The six-manifolds with $SU(3)$ symmetry are quite restricted and can be seen from the fixed point formula to have vanishing Rarita-Schwinger character-valued index. It may be possible to accommodate $SU(2) \times U(1)$ in 10 dimensions and to obtain leptons but not quarks as Rarita-Schwinger zero modes.

Evidently, whatever is the behavior of the Rarita-Schwinger operator on spaces that are not homogeneous, to obtain physics in this way would not be easy.

VI Elementary Gauge Fields

We have so far assumed (except for some brief remarks at the end of the last section) that gauge fields are not elementary but arise as components of the metric tensor in $4 + n$ dimensions. As this assumption has led to difficulties, let us reconsider it. In this section we shall suppose that elementary gauge fields do play a crucial role. Thus we are considering a much less ambitious program. We can no longer hope to unify all forces or calculate all couplings, but we still can hope to explain the fermion quantum numbers; we can try to start with a simple fermion representation in $4 + n$ dimensions and obtain a complicated, repetitive one in 4 dimensions after compactification.

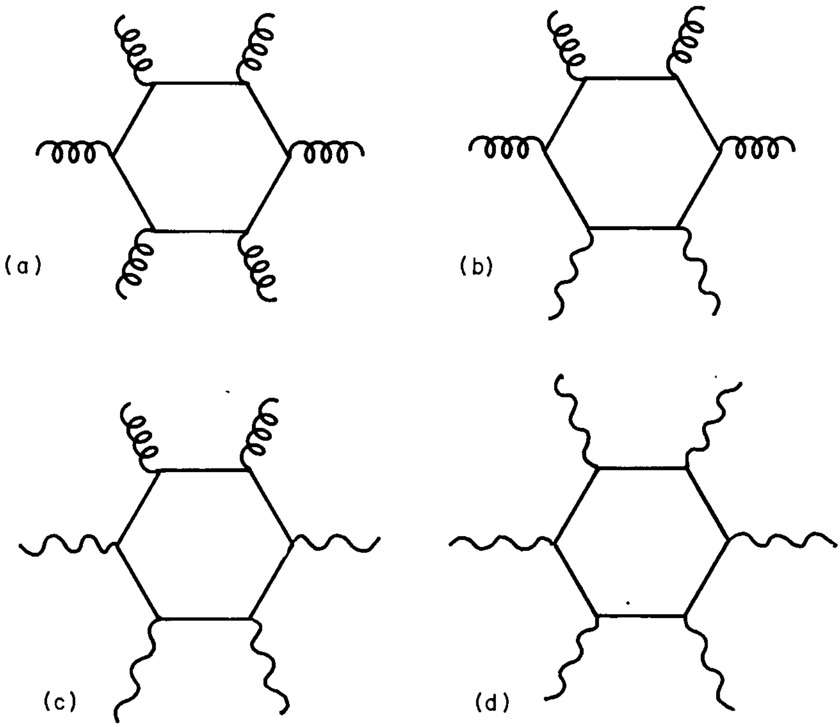

Figure 6
The anomalous diagrams in (say) 10 dimensions. Loopy lines are gravitons; wavy lines are gluons.

We must start in $4 + n$ dimensions with gauge fields that couple differently to left- and right-handed fermions. For if the gauge quantum numbers of the fermions are vectorlike, they will remain vectorlike after any compactification.

Non-vector-like couplings of elementary gauge fields raise an immediate and serious problem of canceling anomalies. In $2p$ dimensions the diagrams with $p + 1$ external gluons, $p - 1$ external gluons and two gravitons, $p - 3$ external gluons and four gravitons, etc., are all anomalous[18] (figure 6). If the gauge field A^a_μ couples to left-handed and right-handed spin $\frac{1}{2}$ fermions via matrices M^a_L and N^a_R, respectively, the condition for anomaly cancellation is

$$\mathrm{Tr}(M^a_L)^r = \mathrm{Tr}(N^a_R)^r, \qquad r = p + 1, p - 1, p - 3, p - 5, \ldots . \tag{51}$$

Equation (51) is trivially obeyed if $M_L^a = N_R^a$, but this is a vectorlike theory that will remain vectorlike after compactification. Nontrivial solutions of (51) are difficult to find because (51) must hold for many values of r.

If charged spin $\frac{3}{2}$ fields are included, the anomaly conditions are more complicated. Anomalous diagrams with four or more external gravitons have different tensor structure for spin $\frac{1}{2}$ and spin $\frac{3}{2}$ fields and must cancel separately. The anomalous diagrams with zero or two external gravitons (and $p + 1$ or $p - 1$ external gluons) could possibly cancel between spin $\frac{1}{2}$ and spin $\frac{3}{2}$, in general, but it is impossible to arrange this in known supergravity theories. We shall not consider models with charged spin $\frac{3}{2}$ fields.

Although it is difficult to find a nontrivial solution of (51), there is a big bonus for doing so. One automatically gets a theory that reduces to an anomaly-free theory in 4 dimensions, because anomaly-free theories remain anomaly-free after any compactification. Cancellation of anomalies is the condition under which the effective action in $4 + n$ dimensions is gauge invariant and generally covariant. If it has these properties in $4 + n$ dimensions, it must retain them after any valid approximation, such as approximate reduction to a 4-dimensional effective action.

A relatively simple solution of (51) is the following. Consider, in $2p$ dimensions, a theory with gauge group $O(2p + 6)$ [or $O(2p + 4k + 6)$, $k \geqslant 0$]. Let the positive chirality spinors of the Lorentz group $O(1, 2p - 1)$ transform as positive chirality spinors of the gauge group $O(2p + 6)$; and let the negative chirality $O(1, 2p - 1)$ spinors transform as negative chirality spinors of $O(2p + 6)$. For $p = 2$ this is the usual $O(10)$ model in 4 dimensions. For any p, this theory is anomaly-free. It obeys (51) because the first $p + 2$ Casimir operators of the left- and right-handed spinor representations of $O(2p + 6)$ are equal.[11] The fermion representation of the $O(2p + 6)$ model in $2p$ dimensions is irreducible in the sense that a combined parity and internal parity operation exchanges the left- and right-handed fermions. This is the simplest theory with nontrivial anomaly cancellation that I can find in $d > 4$.

Despite starting with a non-vector-like theory, we are not assured of keeping this property after dimensional reduction. The familiar relation

$$\Gamma_1 \cdots \Gamma_{4+n} = \Gamma_1 \cdots \Gamma_4 \cdot \Gamma_5 \cdots \Gamma_{4+n} \tag{52}$$

shows that the $(4 + n)$-dimensional chirality operator $\Gamma_1 \cdots \Gamma_{4+n}$ differs from the 4-dimensional operator $\Gamma_1 \cdots \Gamma_4$ by a factor $\Gamma_5 \cdots \Gamma_{4+n}$ that may be

11. This is so because $\mathrm{Tr}\, \sigma_{i_1 j_1} \sigma_{i_2 j_2} \cdots \sigma_{i_k j_k} \cdot \tilde{\Gamma} = 0$ for $k \leqslant p + 2$, if $\tilde{\Gamma}$ is the product of all $2p + 6$ gamma matrices of $O(2p + 6)$ and $\sigma_{ij} = [\gamma_i, \gamma_j]$ are the group generators.

plus or minus one. The non-vector-like nature is washed out after naive dimensional reduction.

Indeed, if the gauge fields have zero expectation value, they are of no help whatsoever in avoiding the problems discussed in previous sections. They simply do not play any role. But vacuum expectation values of gauge fields that can be smoothly turned on or off are likewise irrelevant, for the usual reasons. We therefore must consider gauge field expectation values that cannot be smoothly turned on or off. The prototype of such a thing is the Dirac magnetic monopole, which because it carries topological information and obeys a Dirac quantization condition cannot be smoothly turned on or off. We must place a generalized monopole "inside" the Kaluza-Klein space. Models of this general type have been considered independently by Randjbar-Daemi, Salam, and Strathdee [14].

For illustrative purposes we set $p = 5$ and consider an $O(16)$ theory in 10 dimensions. As explained earlier, the left- (or right-) handed $O(1,9)$ spinors will be chosen as left- (or right-) handed spinors of $O(16)$. We shall consider three models.

In the first two models we take the 6 compact dimensions to be CP^3—the unique 6-dimensional space with $SU(4)$ symmetry. We shall assume that only a single (abelian) component of the $O(16)$ gauge field has an expectation value.

On CP^3 there is a topologically nontrivial $U(1)$ gauge field. [It is a $U(1)$ connection on the basic nontrivial line bundle over CP^3.] This $U(1)$ gauge field can be chosen to be $SU(4)$ invariant in a unique way. Its strength, like the Dirac monopole charge, must be an integer n in certain units.

Our first two models correspond to two ways of embedding $U(1)$ in $O(16)$.

i. The embedding

$$\begin{pmatrix} & 1 & & & & & \\ -1 & & & & & & \\ & & & 1 & & & \\ & & -1 & & & & \\ & & & & \ddots & & \\ & & & & & & 1 \\ & & & & & -1 & \end{pmatrix} \tag{53}$$

breaks $O(16)$ to $SU(8) \times U(1)$. The 4-dimensional gauge group is therefore $SU(4) \times SU(8) \times U(1)$, where $SU(4)$ originates from gravity and $SU(8) \times U(1)$ from $O(16)$.

The quantum numbers of fermion zero modes can be computed using the methods of section III or IV. One finds rather complicated anomaly-free representations. We shall here consider only the minimal case of monopole number $n = 1$. The representation of $SU(4) \times SU(8) \times U(1)$ that emerges is as follows. Let V^i and W^j be the fundamental 4- and 8-dimensional representations of $SU(4)$ and $SU(8)$, respectively. Let S^{ij} be the symmetric product of two V^i, and $\bar{S}_{ij}$ its complex conjugate; and let A^{jk} be the antisymmetric product of two W^j and $\bar{A}_{jk}$ its complex conjugate. Then the left-handed zero modes transform as $(V^i, W^j)^3 \oplus (1, A_{ij})^{-2} \oplus (\bar{S}_{ij}, 1)^{-4}$; the superscript is the $U(1)$ charge. The right-handed massless fermions transform of course in the conjugate representation $(\bar{V}_i, \bar{W}_j)^{-3} \oplus (1, A^{ij})^2 \oplus (S^{ij}, 1)^4$. Despite its complexity, this representation is anomaly-free. For reasons that are not at all clear, this representation is closely related to the supergroup $SU(4|8)$ [30].

ii. We can instead embed $U(1)$ in $O(16)$ as

$$\left(\begin{array}{c|c} \begin{matrix} 0 & & & \\ & \cdot & & \\ & & \cdot & \\ & & & 0 \end{matrix} & \\ \hline & \begin{matrix} & 1 & & & & \\ -1 & & & & & \\ & & & 1 & & \\ & & -1 & & & \\ & & & & & 1 \\ & & & & -1 & \end{matrix} \end{array}\right), \tag{54}$$

breaking $O(16)$ to $O(10) \times SU(3) \times U(1)$. In this case the monopole number must be even, for topological reasons [to ensure proper Dirac quantization for particles in the *spinor* as well as *vector* representations of $O(10)$]. Thus we take the minimal case $n = 2$.

The zero modes can be shown in this case to consist of eight $O(10)$ families (left-handed 16 and right-handed $\overline{16}$) and no antifamilies. Under $SU(4)$

(from gravity) $\times SU(3) \times U(1)$ [from $O(16)$] they transform as follows: they are neutral under $SU(3)$ and transform as $4^3 \oplus \bar{4}^{-3}$ under $SU(4) \times U(1)$.

iii. For our last model, we consider a case in which a nonabelian subgroup of $O(16)$ has a vacuum expectation value. On any 6-dimensional Riemannian manifold there is a natural $O(6)$ gauge field. One simply takes the spin connection $\omega_\mu{}^i{}_j$ and regards it as a gauge field $A_\mu{}^i{}_j$. We embed $O(6)$ in $O(16)$ in the obvious way:

$$\left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & O(6) \end{array}\right) . \tag{55}$$

This breaks $O(16)$ down to $O(10)$.

We shall *not* assume the 6 compact dimensions to be CP^3. The Dirac equation for this system can be analyzed in general. The zero modes are always ordinary families, never antifamilies and never other representations of $O(10)$. The families are neutral under any continuous symmetries of the 6-dimensional space B.

The number of families always equals the Euler characteristic of B. Here are some examples:

$$\begin{array}{ll} S^6: & 2 \text{ families}, \\ S^2 \times S^4: & 4 \text{ families}, \\ S^2 \times S^2 \times S^2: & 8 \text{ families}, \\ CP^3: & 4 \text{ families}, \\ S^3 \times S^3: & 0 \text{ families}. \end{array} \tag{56}$$

In 10 dimensions, the number of families is always even. A similar model in 8 or 12 dimensions can give an odd number of families.

Why is the number of families equal to the Euler characteristic of B? With $A_\mu{}^i{}_j \sim \omega_\mu{}^i{}_j$, the $O(16)$ spinor index behaves as an extra Lorentz spinor index of B. The fermi field is therefore a spinor-spinor, a field with two independent 6-dimensional spinor indices. Such a field is equivalent to the de Rham complex of antisymmetric tensor fields. The Dirac operator becomes the d and d^* operators on differential forms, and the number of zero modes (weighted by chirality) is the Euler characteristic.

VII Other Contexts

The preceding discussion has made clear that in Riemannian geometry it will be very difficult to obtain realistic fermion quantum numbers as zero modes of physically acceptable wave operators. One way out, considered in the last section, is to introduce elementary gauge fields. In the context of Kaluza-Klein theory, this is a rather disappointing possibility. What other alternatives might there be?

We must modify Riemannian geometry in some way. One possibility is that the tangent space group in $4 + n$ dimensions is not $O(1, 3 + n)$ but a smaller group G. (This possibility has been considered by Davidson [30] and in much detail by Weinberg [31].) The smaller group will have more representations, in general; some of the new representations may correspond to new options for the spins of massless particles.

For instance, G may be $O(1,3) \times O(n)$. This group has a representation with spin $\frac{1}{2}$ under $O(1,3)$ and (say) spin $\frac{5}{2}$ under $O(n)$. Since the "true" spin is $\frac{1}{2}$, there are no timelike modes with wrong metric and no difficulty in writing a sensible wave operator. In fact, in the ground state the spin $(\frac{1}{2}, \frac{5}{2})$ field may be regarded as a tensor spinor $\psi_{ij\alpha}$ with i and j constrained to be tangent to the Kaluza-Klein dimensions; a satisfactory wave equation is $\not{D}\psi_{ij} = 0$. (Once one considers fluctuations away from the ground state these formulas do not have a simple generalization, but *some* generalization would emerge in any theory with restricted tangent space group.) Since the spin $\frac{5}{2}$ and higher spin operators in the internal space can readily have complex zero modes (note the discussion of CP^3 in sections V), this would enable us to obtain non-vector-like theories after compactification.

If n is even, $n = 2k$, one could consider a theory with tangent space group $O(1,3) \times U(k)$. The $U(k)$ group of the internal space is the tangent space group in Kahler geometry. In Kahler geometry, one may write many variants of the Dirac equation [corresponding to the many representations of $U(k) \cong SU(k) \times U(1)$ that do not extend to representations of $O(2k)$] that lack analogs in Riemannian geometry. This is the subject of $\bar{\partial}$ cohomology, a major subject in Kahler geometry. The modified Dirac equations of Kahler geometry can readily have complex zero modes. For instance, the equations for a charged spin $\frac{1}{2}$ field interacting with a magnetic monopole on S^2 or CP^3 can be regarded as equations in Kahler geometry; therefore, the model of section three and the first two models of section VI can be viewed in this light.

Of course, it is disappointing to consider tangent space groups like $O(1,3) \times O(n)$ or $O(1,3) \times U(k)$ that are product groups. One would much prefer a unified group, even if smaller than $O(1,3+n)$. However, Weinberg [31] has shown that if one desires Lorentz invariance in 4 dimensions, product groups are the only possibilities.

If one is willing to envisage a product group G, one must still find a sensible equation—replacing the Einstein equation—for the time dependence of the G connection. And presumably one must face at some point the unrenormalizability of quantum field theory in $4 + n$ dimensins, which is likely to persist in this context.

A more drastic modification of Riemannian geometry would be to assume that the underlying theory is not a field theory of the usual kind but a theory of some other type. For instance, at present the supersymmetric string theories in 10 dimensions [7] would appear to be very attractive candidates—especially the $n = 2$ theory, which is chirally asymmetric and anomaly-free. This theory naively reduces at low energies to 10-dimensional supergravity, but unlike that theory [7, 32], it is likely to be a finite theory to all orders.

Naive compactification of the string theory proceeds via 10-dimensional field theory and suffers from the problems of 10-dimensional field theory in describing fermion quantum numbers. However, there may be "inherently stringy" ways to compactify the string theory directly to 4 dimensions, without 10-dimensional field theory as an intermediate stage. The rules and problems of Riemannian geometry might not apply in such a case—though I hope some of the concept of this paper would be relevant. I would consider this the most attractive possibility, but unfortunately with the present incomplete understanding of the string theory, it is difficult to pursue this possibility.

I shall, however, discuss one aspect of the problem. The string theory has a single dimensionless coupling constant λ. (It is essentially Newton's constant written in units of the Regge slope.) It is generally believed that λ is an arbitrarily adjustable constant. If so, the fermion quantum numbers cannot depend on λ. I believe that instead, when the string theory is more fully understood, it will be seen that (with proper normalization of λ) the mathematical consistency of the theory will require that $1/\lambda$ be an integer. The action of the string field theory that has been partly constructed [33] is rather analogous to the effective action of the large N expansion in QCD. We now know [34] that the large N effective action is multivalued, defined only modulo $2\pi N$. This means that, internal to the $1/N$ expansion, N must be an integer; this is analogous to the quantization of coupling constants in some $(2 + 1)$-dimensional field theories [35]. The close analogy between λ and $1/N$ strongly suggests that the string field theory action is likewise multivalued, defined only modulo $2\pi/\lambda$. This would mean that $1/\lambda$ would have to be an integer n; the proper quark and lepton quantum numbers might emerge only for a definite value of n.

VIII Massless Scalars

The main focus of this paper is the question of obtaining massless fermions as zero modes in Kaluza-Klein theory. However, massless bosons are also important. Massless spin 1 and spin 2 bosons have a well-known origin in Kaluza-Klein theory, and arise for a simple reason, as reflections of unbroken local symmetries. Massless scalars do not have such a simple rationale. Yet a good explanation of the existence of massless charged scalars would be of utmost importance: it would offer a solution [36] to the problem of the existence of widely disparate mass scales in physics.

Any bose field of a Kaluza-Klein theory might have modes that would be seen as charged scalars in 4 dimensions. We must consider then the gravitational field, the antisymmetric tensor fields of certain supergravity and other theories, and gauge fields. [More generally, in suitable backgrounds, different fields may mix; mixed modes may be considered under (i) or (iii), to follow.]

In general terms, we do not want scalars that are massless for reasons of symmetry. The only scalars kept massless by any symmetry argument of the usual sort are Goldstone bosons, which are always neutral under any unbroken gauge symmetries and hence are no help in solving the hierarchy problem. Moreover, for a Goldstone boson any potential at all is forbidden; there is no reason for a Goldstone boson field to acquire tiny but nonzero vacuum expectation values. We wish a more subtle argument for bosonic zero modes, perhaps a topological argument, which will forbid mass terms but allow quartic self-couplings.

Let us consider in turn the cases of gravitational, antisymmetric tensor, and Yang-Mills zero modes.

i. It seems that very little is known about the conditions under which some oscillations in the geometry of a compact space B will correspond to massless scalars. If B is Ricci flat, $R_{\mu\nu} = 0$, a "breathing mode" in which the geometry of B is uniformly dilated corresponds to a massless scalar, because the equation $R_{\mu\nu} = 0$ has a scaling symmetry and does not determine the radius of B. This mode is always neutral under continuous symmetries, so is not helpful in solving the hirarchy problem. The scaling symmetry of the classical equation $R_{\mu\nu} = 0$ is not a symmetry of quantum Kaluza-Klein theories, and therefore [37] (unless there is an unbroken supersymmetry) these modes get nonzero mass at the one-loop level.

In models with an unbroken supersymmetry at the tree level, there are some known cases in which some oscillations in the metric of B correspond to charged massless scalars at the tree level.[12] Little is known about the possibility of eventual supersymmetry breaking in these models.

ii. Many supergravity theories contain antisymmetric tensor fields—for instance, the third-rank antisymmetric tensor field A_{ijk} of 11-dimensional supergravity. The Lagrangian is constructed from the curl of A, $F_{ijkl} = \partial_i A_{jkl} \pm$ cyclic permutations. This curl is invariant under the gauge transformation $A_{ijk} \to A_{ijk} + (\partial_i \Lambda_{jk} +$ cyclic permutations$)$. The Lagrangian for a kth rank antisymmetric tensor gauge field is

$$\mathscr{L} = \frac{1}{2(k+1)!} \int d^n x \, (F_{i_1 \cdots i_{k+1}})^2. \tag{57}$$

The field equations derived from this Lagrangian may readily have zero modes for topological reasons.

The physical interpretation of these modes depends on how many indices of $A_{i_1 \cdots i_k}$ are tangent to B and how many are tangent to the space-time directions. If all indices are tangent to B, we get a massless scalar in 4 dimensions. If all indices but one are tangent to B, we get a massless spin 1 particle in 4 dimensions. If all indices but two are tangent to B, we get a massless antisymmetric tensor in 4 dimensions that again describes a massless scalar. Other cases do not give rise to propagating modes in 4 dimensions.

In general, the number of zero modes (modulo gauge transformations) of $A_{i_1 \cdots i_k}$ with q indices tangent to B is equal to a topological invariant known as the qth Betti number of B. As an example, one may consider in 11 dimensions the spaces M^{pqr} with $SU(3) \times SU(2) \times U(1)$ symmetry [4]. For most of these spaces, the first and third Betti numbers vanish, so one does

12. I thank M. Duff for a discussion of this point.

not get massless scalars. (The exceptions are $CP^2 \times S^3$ and $CP^2 \times S^3/Z^k$, for which the third Betti number is one and one gets one massless scalar in 4 dimensions; and $CP^2 \times S^2 \times S^1$, for which the first and third Betti numbers are one and one gets two massless scalars.) However, for the M^{pqr} the second Betti number is one (except for $CP^2 \times S^2 \times S^1$, where it is two), so one would get one (or two) massless spin one particles in 4 dimensions [in addition to the gauge fields of $SU(3) \times SU(2) \times U(1)$ coming from the metric tensor]. These massless spin 1 particles do not have minimal couplings to any matter fields. They interact through derivative couplings, such as magnetic moment couplings to fermi fields. They would give rise to long-range spin-spin forces of roughly gravitational strength; presumably this is far too weak to be detectable.

There is, however, an old theorem that zero modes of antisymmetric tensor fields are always neutral under any continuous symmetries. Essentially, this is true because, by the de Rham-Hodge theory, zero modes of the qth antisymmetric tensor field on B correspond to topological classes of closed q-dimensional submanifolds of B. A continuous symmetry cannot change the topological class of a submanifold, so it leaves invariant all of the zero modes of antisymmetric tensor fields.

Here is an analytical proof.[13] Let d be the curl operator, so the curl of the antisymmetric tensor field A will be denoted $F = dA$. [Thus, $(dA)_{i_1 \cdots i_{k+1}} = (\partial_{i_1} A_{i_2 \cdots i_{k+1}} \pm \text{cyclic permutations})$.] The change in A under a gauge transformation is $A \to A + d\Lambda$, where Λ is an antisymmetric tensor field with one less index than A. What is a massless mode of the A field? Setting the momentum in the Minkowski directions to zero, we calculate the energy (per unit volume) of an antisymmetric tensor field A by integrating over the compact dimensions. From (57), the integral is $\int_B d\phi \, (dA)^2$, so a massless mode is an antisymmetric tensor field defined on B such that $dA = 0$. Actually, we want zero modes that cannot be gauged away, so we want solutions of $dA = 0$ modulo gauge transformations $A \to A + d\Lambda$. (Since $d^2 = 0$, any pure gauge $A = d\Lambda$ obeys $dA = 0$.)

Now let $K^i(\phi^j)$ be an arbitrary Killing vector field, generating the infinitesimal symmetry transformation $\phi^i \to \phi^i + \varepsilon K^i(\phi^j)$. To show that the zero modes of A are neutral under arbitrary continuous symmetries, we must show that the transformation generated by K^i leaves A unchanged, or more exactly that it leaves A unchanged up to a gauge transformation.

13. See, for instance, reference [38] for further material on antisymmetric tensor fields.

The generator of the symmetry is known as the Lie derivative. Acting on antisymmetric tensor fields, it takes a particularly simple form and can be conveniently defined as follows. Let i_K be the operation of contraction with K; explicitly, for any antisymmetric tensor field $A_{i_1 \cdots i_k}$, $i_K A$ is an antisymmetric tensor field with one index less: $(i_K A)_{i_1 \cdots i_{k-1}} = K^{i_0} A_{i_0 i_1 \cdots i_{k-1}}$. Then $\mathscr{L}_K$, on differential forms, can be defined as follows:

$$\mathscr{L}_K = d i_K + i_K d. \tag{58}$$

Thus, the infinitesimal change of A under the transformation generated by K is $A \to A + \varepsilon \mathscr{L}_K A$. Equation (58), which may appear unfamiliar, can be seen to agree with the formulas in books on general relativity.

What we must show is that acting on any zero mode, $\mathscr{L}_K$ vanishes up to a gauge transformation. This is not difficult. A zero mode A obeys $dA = 0$, so $\mathscr{L}_K A = (d i_K + i_K d)A = d(i_K A)$, but $d(i_K A)$ is the change in A under a gauge transformation (with gauge parameter $\Lambda = i_K A$).

The Atiyah-Hirzebruch theorem, which is more delicate, may be seen as a generalization of this classical result to the spin $\frac{1}{2}$ case.

iii. Now we consider zero modes of gauge fields. Abelian gauge fields are a special case of the antisymmetric tensor fields just considered, but elementary nonabelian gauge fields $A_i{}^a$ that might be present in the $(4 + n)$-dimensional theory raise different issues.

Zero modes of $A_i{}^a$ in which the vector index i is tangent to the compact dimensions will be observed as massless scalars in 4 dimensions. It is certainly possible—classically—to obtain charged massless scalars in this way. The simplest example is the original Kaluza-Klein theory, the sole extra dimension being a circle S^1. If elementary nonabelian gauge fields are posited, then the mode in which $A_5{}^a$ is a constant, independent of position on the circle, gives rise in 4 dimensions to a multiplet of massless scalars in the adjoint representation.

In this case, there are (classically) no quartic interactions, but that is not a general property. In $4 + n$ dimensions, if the compact dimensions are circles $S^1 \times \cdots \times S^1$, then the constant modes of all $A_i{}^a$, $i = 5, \ldots, 4 + n$, are massless scalars, and one obtains at the tree level the quartic potential

$$V(A_i{}^a) = \frac{e^2}{4} \sum_{i,j=5,\ldots,4+n} \operatorname{Tr}[A_i, A_j]^2. \tag{59}$$

There are other, more elaborate scenarios that lead at the classical level to charged massless scalars as zero modes of Yang-Mills fields. Here is one such scenario. Consider an 8-dimensional world in which the 4 extra dimensions are $S^2 \times S^2$, giving rise to an $SU(2) \times SU(2)$ symmetry. Let there be an elementary $SU(2)$ gauge field, one of whose components has an expectation value, breaking $SU(2)$ to $U(1)$. The total unbroken gauge group in 4 dimensions is then $SU(2) \times SU(2) \times U(1)$. Suppose that the gauge field expectation value on $S^2 \times S^2$ is of the monopole type, with equal monopole number p on each two-sphere. (According to Dirac, p is an integer or a half-integer.) Then, if $p \geqslant 1$, it can be shown that for topological reasons the Yang-Mills equations have massless zero modes at the tree level; they have $U(1)$ charge ± 1 and transform under $SU(2) \times SU(2)$ as $(p, p-1) \oplus (p-1, p)$. Although these modes are massless, they have at the tree level a nontrivial quartic potential.

What these examples have in common is that in all examples I am aware of, there is no reason for the massless modes to remain massless when loop corrections are considered. Indeed, in the first example considered (the massless mode being a constant on the circle), a one-loop calculation has been carried out in the abelian case[37], showing that a nonzero mass does arise unless there is a bose-fermi cancellation.

IX The Cosmological Constant

Until now we have considered exclusively the zero modes of wave operators. However, the spirit of this paper is to study qualitative problems that might be solved without full understanding of the details of a Kaluza-Klein theory. In that spirit, we shall here consider another qualitative problem of outstanding significance: the apparent vanishing in 4 dimensions of the cosmological constant.

There has been much interest in recent years[39] in the possibility of a dynamical explanation of the vanishing of the cosmological constant—the possibility of a theory in which regardless of the value of the bare parameters, the cosmological constant spontaneously relaxes to zero. We want, in other words, a mechanism analogous to the axion mechanism for avoiding strong CP violation. Some ideas in this section have been introduced independently in work cited in reference [39].

We would like to find a theory in which the classical equations do not determine the effective cosmological constant—the actual, macroscopic curvature of 4-dimensional space. The classical equations should admit for any values of the bare parameters a one-parameter family of solutions, depending on an integration constant. The effective cosmological constant should depend on this integration constant. We shall look for a mechanism by which the integration constant spontaneously relaxes to the value at which space-time is macroscopically flat.

Of course, it is easy to find a theory in which there is an undetermined integration constant at the classical level. Consider a theory of scalar fields ϕ_i. If the potential energy $V(\phi_i)$ is independent of one of these fields ϕ, the vacuum expectation value of that field will be undetermined at the classical level. However, precisely because $V(\phi^i)$ is independent of ϕ, the effective cosmological constant will be independent of ϕ.

To find a theory with an undetermined integration constant *upon which the cosmological constant depends* requires a different approach. The only way I know to do this in 3 + 1 dimensions is to introduce a third-rank antisymmetric tensor gauge field $A_{\mu\nu\alpha}$. As in section VIII, the Lagrangian is

$$\mathscr{L} = -\frac{1}{48}\int d^4x\,(F_{\mu\nu\alpha\beta})^2, \tag{60}$$

where $F_{\mu\nu\alpha\beta}$ is the gauge invariant curl, $F_{\mu\nu\alpha\beta} = (\partial_\mu A_{\nu\alpha\beta} \pm \text{cyclic permutations})$. If we define the scalar $F = (1/24)\varepsilon^{\mu\nu\alpha\beta}F_{\mu\nu\alpha\beta}$ ($\varepsilon^{\mu\nu\alpha\beta}$ being the 4-dimensional Levi-Civita symbol), then the equation of motion from (60) is $\partial_\mu F = 0$. Thus, F is a constant—but the constant is a constant of integration, not determined by the classical equations. And the cosmological constant definitely depends on F. It equals its value at $F = 0$, plus $F^2/8$.

However, this example is too trivial. The equation of motion $\partial_\mu F = 0$—which is an exact statement, even quantum mechanically—appears to tell us that the integration constant F cannot possibly relax to the value at which the effective cosmological constant would vanish.

A less trivial example of the same kind arises if the third-rank antisymmetric tensor field is considered in $4 + n$ dimensions. We consider the Lagrangian

$$\mathscr{L} = \int d^{4+n}x\,\sqrt{g}\left(\frac{1}{16\pi G}R - \frac{1}{48}(F_{\mu\nu\alpha\beta})^2 - \Lambda_0\right). \tag{61}$$

For $n = 7$, this differs from the bosonic part of 11-dimensional supergravity by the omission of an FFA term (its inclusion would not affect our discussion) and the inclusion of a nonzero bare cosmological constant Λ_0 (forbidden by supersymmetry in 11 dimensions but needed in our discussion to get a solution in which 4 dimensions are flat). The solution we will discuss is the Freund-Rubin solution[40], generalized to $\Lambda_0 \neq 0$.

We look for a solution of the classical equations derived from (61) in which space-time takes the form $D^4(\lambda) \times S^7(R)$, where $D^4(\lambda)$ is a 4-dimensional de Sitter space of positive, negative, or zero curvature λ, and $S^7(R)$ is a seven-sphere of radius R.

In looking for such a solution, we encounter—as Freund and Rubin did—the possibility of a nonzero value of $F = F_{0123}$. As in the 4-dimensional case, the equations determine only the *derivatives* of F, not F itself. One may assume an arbitrary value of F, and use the classical equations to solve for λ and R in terms of F. For a whole continuous range of the bare parameters in (61), there is a value of F at which λ—the curvature of ordinary space—vanishes.

The gain in going from 4 to $4 + n$ dimensions is that in $4 + n$ dimensions $F_{\mu\nu\alpha\beta}$ has a nontrivial dynamics with propagating modes as well as an integration constant. One can at least imagine that there may be a quantum mechanical mechanism by which the integration constant F spontaneously relaxes to some special value—one hopes the value at which $\lambda = 0$. But what might this mechanism be?

In condensed matter physics there are some fascinating systems with the following properties. (For recent theoretical discussions and references to previous work see reference [41].) The macroscopic equations have a one-parameter family of solutions depending on an integration constant x. There is a critical value of x, say $x = x_0$, such that the classical solution is stable against small oscillations for $x \geqslant x_0$ and unstable for $x < x_0$.

What value of x would be observed physically? One would hardly expect to observe the unstable solutions of $x < x_0$, but one might expect that, depending on initial conditions, any stable solution with $x \geqslant x_0$ would be accessible.

The surprise is that it is claimed[41] that a whole class of systems spontaneously relaxes—by means that are not well understood—to the threshold of stability, $x = x_0$. The fact that the mechanism is so little understood in the condensed matter context invites the speculation that a similar phenomenon could occur in the case of the cosmological constant. Although $\lambda = 0$ (flat space) is not exactly a threshold of stability in any obvious sense, it is certainly the dividing point between two qualitatively different regimes, de Sitter space and anti-de Sitter space. Anti-de Sitter space has a positive energy theorem, which de Sitter space does not[42]; but de Sitter space has a global initial value hypersurface, which anti-de Sitter space does not[43]. They are certainly very different. Perhaps the little-understood mechanism by which the condensed matter systems relax has a "cosmological" analog—though it is not yet clear whether it is de Sitter or anti-de Sitter space that should correspond in the analogy to $x > x_0$.

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*Editors' note: The paper read to the conference by Hawking appears in this volume.