ARITHMETIC RAMSEY THEORY
PROBLEM SET # 3

(1) Let $M \geq 1$. Recall that we defined $\mu_M : \mathbf{Z} \to [0,1]$ by
$$\mu_M(h) := \frac{\#\{(h_1,h_2) \in [M]^2 : h_1 - h_2 = h\}}{M^2}.$$
Prove that
  (a) $\operatorname{supp} \mu_M \subset (-M, M)$,
  (b) $\|\mu_M\|_{\ell^1} = 1$,
  (c) $\|\mu_M\|_{\ell^2} \leq \frac{1}{M}$, and
  (d) $\int_{\mathbf{T}} |\widehat{\mu_M}(\xi)| d\xi = \frac{1}{M}$.

(2) Prove that for any permutation σ of $[s]$,
$$\Delta_{(h_1,h_1'),\dots,(h_s,h_s')} f = \Delta_{(h_{\sigma(1)},h_{\sigma(1)}'),\dots,(h_{\sigma(s)},h_{\sigma(s)}')} f.$$

(3) Prove that the quantity
$$\frac{1}{N} \sum_{x \in \mathbf{Z}} \mathbf{E}_{\substack{h_i,h_i' \in H_i \\ i \in [s]}} \Delta_{(h_1,h_1'),\dots,(h_s,h_s')} f(x)$$
appearing in the definition of the Gowers box norms is always real and nonnegative.

(4) Verify that
$$\|f\|_{\square^s_{H_1,\dots,H_s}(N)}^{2^s} = \frac{1}{N} \sum_{x \in \mathbf{Z}} \mathbf{E}_{\substack{h_i,h_i' \in H_i \\ i \in [s]}} \prod_{\omega \in \{0,1\}^s} \mathcal{C}^{|\omega|} f(x + \mathbf{h} \cdot \omega + \mathbf{h'} \cdot (\mathbf{1} - \omega)),$$
where $|\omega|$ denotes the number of 1's appearing in ω, $\mathbf{h} = (h_1, \dots, h_s)$, $\mathbf{h'} = (h_1', \dots, h_s')$, and $\mathbf{1}$ denotes the vector of all 1's.

(5) (a) Prove the *Gowers–Cauchy–Schwarz inequality*: Let $s, N \in \mathbf{N}$, $H_1, \dots, H_s \subset \mathbf{Z}$ be finite and nonempty, and, for each $\omega \in \{0,1\}^s$, $f_\omega : \mathbf{Z} \to \mathbf{C}$ be finitely supported. Then,
$$\left| \frac{1}{N} \sum_{x \in \mathbf{Z}} \mathbf{E}_{\substack{h_i,h_i' \in H_i \\ i \in [s]}} \prod_{\omega \in \{0,1\}^s} \mathcal{C}^{|\omega|} f_\omega(x + \mathbf{h} \cdot \omega + \mathbf{h'} \cdot (\mathbf{1} - \omega)) \right| \leq \prod_{\omega \in \{0,1\}^s} \|f_\omega\|_{\square^s_{H_1,\dots,H_s}(N)}.$$
(Hint: Proceed by induction using the Cauchy–Schwarz inequality.)
  (b) Prove that $\|\cdot\|_{\square^s_{H_1,\dots,H_s}(N)}$ satisfies the triangle inequality. (Hint: Raise both sides to the 2^s-th power and apply the Gowers–Cauchy–Schwarz inequality.)
  (c) Deduce that $\|\cdot\|_{\square^1_{H_1}(N)}$ is a seminorm, and then prove that $\|\cdot\|_{\square^s_{H_1,\dots,H_s}(N)}$ is a genuine norm whenever $s \geq 2$.

(6) Prove van der Corput's inequality: Let $N > H > 0$ and $f : \mathbf{Z} \to \mathbf{C}$. Then,

$$\left|\mathbf{E}_{x\in[N]} f(x)\right|^2 \leq \frac{N+H}{N} \mathbf{E}_h^{\mu_H} \left(\frac{1}{N} \sum_{x \in [N]\cap([N]-h)} f(x)\overline{f}(x+h) \right).$$

(Hint: Write

$$\mathbf{E}_{x\in[N]} f(x) = \frac{1}{N}\sum_{x\in\mathbf{Z}} 1_{[N]}(x) f(x) = \frac{1}{N}\sum_{x\in\mathbf{Z}} \mathbf{E}_{h\in[H]} 1_{[N]}(x+h) f(x+h)$$

then apply the Cauchy–Schwarz inequality and make a change of variables.)

(7) Let $N, M, \delta > 0$ and $a_1, \dots, a_m \in \mathbf{Z}$ be nonzero integers. Let $f_0, \dots, f_m : \mathbf{Z} \to \mathbf{C}$ be 1-bounded functions supported on $[N]$. Prove that if

$$\left| \frac{1}{N} \sum_{x\in\mathbf{Z}} \mathbf{E}_{y\in[M]} f_0(x) f_1(x + a_1 x) \cdots f_m(x + a_m x) \right| \geq \delta,$$

then, for all $\delta' \ll_m \delta^{O_m(1)}$, one has

$$\|f_m\|_{\square^m_{H_0,\dots,H_{m-1}}(N)} \gg_m \delta,$$

where $H_0 = a_m[\delta' M]$ and $H_i = (a_m - a_i)[\delta' M]$ for each $i \in [m-1]$.