ARITHMETIC RAMSEY THEORY
PROBLEM SET # 2

Problems marked with a $\star$ are harder.

(1) (a) Mimicking the argument from the lecture, prove that if $A \subset [N]$ contains no nontrivial copies of $x, x + y^3$, then

$$|A| \ll \frac{N}{(\log \log N)^c}$$

for some absolute constant $c > 0$.

(b) ($\star$) Do the same for the progression $x, x + y^2 + y$.

(2) (a) Let $k \geq 3$ and assume that $N \geq k^2$. Prove that if $A \subset [N]$ has density greater than $\frac{k-1}{k}$, then A contains a nontrivial k-term arithmetic progression.

(b) Consider the following density-increment lemma, whose proof is beyond the scope of this course:

Lemma 1. *Fix $k \in \mathbf{N}$ with $k \geq 3$. There exist constants $c_1, c_2 > 0$ such that the following holds. Let $A \subset [N]$ have density $\alpha \leq \frac{k-1}{k}$ and suppose that A contains no nontrivial k-term arithmetic progressions. Then, either $N < \alpha^{-c_1}$, or there exists an arithmetic progression $P = \{a + qx : x \in [N']\}$ with $N' \geq N^{\alpha^{c_1}}$ such that*

$$\frac{|A \cap P|}{|P|} \geq \alpha + \alpha^{c_2}.$$

Assuming this lemma, prove that if $A \subset [N]$ contains no nontrivial k-term arithmetic progressions, then $|A| \ll \frac{N}{(\log \log N)^c}$ for some constant $c > 0$.

(c) Suppose one knew the following density-increment lemma:

Lemma 2. *Let $A \subset [N]$ have density α and suppose that A contains no nontrivial nonlinear Roth configurations. Then, either $N \leq 100\alpha^{-100}$, or there exists an arithmetic progression $P = \{a + qx : x \in [N']\}$ with $N' \geq N^{1/4}$ such that*

$$\frac{|A \cap P|}{|P|} \geq \alpha + \alpha^2.$$

Can one prove the nonlinear Roth theorem using this lemma? If so, prove it. If not, explain what goes wrong.

(3) Let $f, g : \mathbf{Z} \to \mathbf{C}$ be finitely supported. Prove the following standard properties of the Fourier transform:

(a) $\widehat{f * g} = \widehat{f} \cdot \widehat{g}$

(b) $f(x) = \int_{\mathbf{T}} \widehat{f}(\xi) e(\xi x) d\xi$ for all $x \in \mathbf{Z}$

(c) $\sum_{x \in \mathbf{Z}} f(x)\overline{g(x)} = \int_{\mathbf{T}} \widehat{f}(\xi)\overline{\widehat{g(\xi)}} d\xi$

(4) (a) Prove that

$$\left| \sum_{N \le n \le N+M} e(\alpha n) \right| \ll \min\left(M, \frac{1}{\|\alpha\|}\right)$$

for all $\alpha, N, M \in \mathbf{R}$ with $M \ge 1$.

(b) Assume that there exists a subset S of $\{-N, \dots, N\}$ of size N' such that $\|n\xi\| \le \varepsilon$ for all $n \in S$. Prove that if $N' \ge 2 + 2\varepsilon N$, then there exists a natural number $q \ll \frac{N}{N'}$ such that $\|q\xi\| \ll \frac{\varepsilon}{N'}$. (Hint: Apply the pigeonhole principle to find $n_1, n_2 \in S$ such that $\|q\xi\| \le \frac{2\varepsilon}{N'-1}$. This gives a good rational approximation $\frac{a}{q}$ to ξ. Use this to show that S must lie in a small number of congruence classes modulo q, and so q must be small.)

(c) Let $\xi \in \mathbf{T}$. Prove that if

$$\left| \mathbf{E}_{n \in [N]} e(\xi n^2) \right| \ge \gamma,$$

then there exists a natural number $q \ll \gamma^{-O(1)}$ such that $\|q\xi\| \ll \frac{\gamma^{-O(1)}}{N^2}$. (Hint: Square $\left|\mathbf{E}_{n \in [N]} e(\xi n^2)\right|$, make a change of variables, and apply the first and second parts of the problem.)