6. Control by Gowers norms

We will also require a bit more notation. Let $\mathcal{C} : \mathbf{C} \to \mathbf{C}$ denote the complex conjugation operator. If $\mu : \mathbf{Z} \to [0, \infty)$ is a finitely supported weight function and $f : \mathbf{Z} \to \mathbf{C}$, we write

$$\mathbf{E}^{\mu}_{x} f(x) := \sum_{x \in \mathbf{Z}} f(x)\mu(x).$$

We normalize the ℓ^p-norms using counting measure: $\|f\|_{\ell^p}^p = \sum_{x \in \mathbf{Z}} |f(x)|^p$ for any $1 \leq p < \infty$. For any $M \geq 1$, we define the *Fejér kernel* weight $\mu_M : \mathbf{Z} \to [0, 1]$ by

$$\mu_M(h) := \frac{\#\{(h_1, h_2) \in [M]^2 : h_1 - h_2 = h\}}{M^2}.$$

Lemma 4 (Basic properties of the Fejer kernel)**.** *Let $M \geq 1$. We have*

(1) $\operatorname{supp} \mu_M \subset (-M, M)$,
(2) $\|\mu_M\|_{\ell^1} = 1$,
(3) $\|\mu_M\|_{\ell^2} \leq \frac{1}{M}$, *and*
(4) $\int_{\mathbf{T}} |\widehat{\mu_M}(\xi)| d\xi = \frac{1}{M}$.

Exercise 12. Prove the above lemma.

Definition 1 (Discrete derivatives)**.** *Let $f : \mathbf{Z} \to \mathbf{C}$ and $h, h' \in \mathbf{Z}$. We define the multiplicative discrete derivative $\Delta_{(h,h')} f : \mathbf{Z} \to \mathbf{C}$ by*

$$\Delta_{(h,h')} f(x) = f(x + h)\overline{f(x + h')}.$$

For any $h_1, h_1', \ldots, h_s, h_s' \in \mathbf{Z}$, we denote the s-fold multiplicative discrete derivative of f by

$$\Delta_{(h_1,h_1'),\ldots,(h_s,h_s')} f = \Delta_{(h_1,h_1')} \cdots \Delta_{(h_s,h_S')} f.$$

Exercise 13. Prove that for any permutation σ of $[s]$,

$$\Delta_{(h_1,h_1'),\ldots,(h_s,h_s')} f = \Delta_{(h_{\sigma(1)},h'_{\sigma(1)}),\ldots,(h_{\sigma(s)},h'_{\sigma(s)})} f.$$

Now, we can define the Gowers box and uniformity norms. The latter (in a slightly different form) were originally introduced by Gowers in his proof of Szemerédi's theorem, and the former are a generalization of these norms that are relevant to the study of nonlinear polynomial progressions.

Definition 2. *Let $s, N \in \mathbf{N}$, $H_1, \ldots, H_s \subset \mathbf{Z}$ be finite and nonempty, and $f : \mathbf{Z} \to \mathbf{C}$ be finitely supported. The Gowers box norm of f with respect to $H_1, \ldots, H_s$ normalized by N is*

$$\|f\|_{\square^s_{H_1,\ldots,H_s}(N)}^{2^s} := \frac{1}{N} \sum_{x \in \mathbf{Z}} \mathbf{E}_{\substack{h_i, h_i' \in H_i \\ i \in [s]}} \Delta_{(h_1,h_1'),\ldots,(h_s,h_s')} f(x).$$

When $H_1 = \cdots = H_s = H$, this is the Gowers U^s-norm of f with respect to H, and is denoted by

$$\|f\|_{U^s_H(N)} := \|f\|_{\square^s_{H,\ldots,H}(N)}.$$

For example,

$$\|f\|_{U^2_H(N)}^4 = \frac{1}{N}\sum_{x\in\mathbf{Z}} \mathbf{E}_{h,h',k,k'\in H} f(x+h+k)\overline{f(x+h+k')f(x+h'+k)}f(x+h'+k').$$

Exercise 14. Verify that the right-hand side above is real and nonnegative, and thus that the definition makes sense.

Exercise 15. Verify that these are, indeed, norms when $s \geq 2$ and, when $s = 1$, a seminorm.

Note that, for any $1 \leq t \leq s-1$, the Gowers box norms satisfy the following recursive identity:

$$\|f\|_{\square^s_{H_1,\dots,H_s}(N)}^{2^s} = \mathbf{E}_{\substack{h_i,h_i'\in H_i\\ i\in[t]}} \|\Delta_{(h_1,h_1'),\dots,(h_t,h_t')} f\|_{\square^{s-t}_{H_{t+1},\dots,H_s}(N)}^{2^{s-t}}.$$

We will shortly see that the U^2-norm of f is essentially the L^4-norm of the Fourier transform of f. Higher-order Fourier analysis is the study of the U^s- and $\square^3$-norms when $s \geq 3$. Gowers proved using repeated applications of the Cauchy–Schwarz inequality that the count of k-term arithmetic progressions in a subset of $[N]$ is controlled by the U^{k-1}-norm. When studying more general polynomial progressions in the integers such as the nonlinear Roth configuration, our starting variables x and y live on different scales, and thus we will need to use van der Corput's inequality instead of just the Cauchy–Schwarz inequality.

Lemma 5 (van der Corput's inequality)**.** *Let $N > H > 0$ and $f : \mathbf{Z} \to \mathbf{C}$. Then, we have*

$$\left|\mathbf{E}_{x\in[N]} f(x)\right|^2 \leq \frac{N+H}{N}\mathbf{E}_h^{\mu_H}\left(\frac{1}{N}\sum_{x\in[N]\cap([N]-h)} f(x)\overline{f}(x+h)\right).$$

Exercise 16. Prove van der Corput's inequality.

For any $N, M > 0$, define

$$\Lambda_3(f,g,h) := \frac{1}{N}\sum_{x\in\mathbf{Z}} \mathbf{E}_{y\in[M]} f(x)g(x+y)h(x+2y).$$

The most basic instance of the Gowers uniformity norms controlling the count of arithmetic progressions is that the U^2-norm controls the count of three-term arithmetic progressions in a set.

Lemma 6 (U^2-control of three-term arithmetic progressions)**.** *Let $N, M, \delta > 0$ and $f, g, h : \mathbf{Z} \to \mathbf{C}$ be 1-bounded functions supported on $[N]$. If*

$$|\Lambda_3(f,g,h)| \geq \delta,$$

then, for all $\delta' \leq \frac{\delta^4}{64}$, we have

$$\|h\|_{U^2_{[\delta' M]}(N)} \gg \delta.$$

Proof. We prove this by repeated applications of the Cauchy–Schwarz and van der Corput inequalities. The assumption is that

$$\left|\frac{1}{N}\sum_{x\in\mathbf{Z}} f(x)\mathbf{E}_{y\in[M]}g(x+y)h(x+2y)\right| \geq \delta.$$

So, since f is 1-bounded and supported on $[N]$, we have by the Cauchy–Schwarz inequality that

$$\frac{1}{N}\sum_{x\in\mathbf{Z}} \left|\mathbf{E}_{y\in[M]}g(x+y)h(x+2y)\right|^2 \geq \delta^2.$$

Now, we apply van der Corput's inequality in the above with $H = \delta' M$, which yields that

$$\frac{1}{N}\sum_{x\in\mathbf{Z}}(1+\delta')\mathbf{E}_h^{\mu_{\delta' M}}\frac{1}{M}\sum_{y\in[M]\cap([M]-h)} g(x+y)\overline{g}(x+y+h)h(x+2y)\overline{h}(x+2y+2h)$$

is at least δ^2. Since $\operatorname{supp}\mu_{\delta' M} \subset (-\delta' M, \delta' M)$, f is 1-bounded, $\|\mu_{\delta' M}\|_{\ell^1} = 1$, and $\delta' \leq \frac{\delta^2}{4}$, for each $h \in \mathbf{Z}$ for which $\mu_{\delta' M}(h) \neq 0$, the sum over $[M]\cap([M]-h)$ can be extended to a sum over all of $[M]$ at the cost of an error of size at most $\frac{\delta^2}{2}$. Thus, using that $(1+\delta') < 2$ and then expanding the definition of $\mu_{\delta' M}(h)$ and making the change of variables $x \mapsto x + h'$, we obtain

$$\mathbf{E}_{h,h'\in[\delta' M]}\frac{1}{N}\sum_{x\in\mathbf{Z}}\mathbf{E}_{y\in[M]}\Delta_{(h,h')}g(x+y)\Delta_{(2h,2h')}h(x+2y) \geq \frac{\delta^2}{4}.$$

Making the change of variables $x \mapsto x - y$ then yields

$$\mathbf{E}_{h,h'\in[\delta' M]}\frac{1}{N}\sum_{x\in\mathbf{Z}}\mathbf{E}_{y\in[M]}\Delta_{(h,h')}g(x)\Delta_{(2h,2h')}h(x+y) \geq \frac{\delta^2}{4}.$$

Arguing analogously with another application of the Cauchy–Schwarz and van der Corput inequalities yields

$$\mathbf{E}_{h,h',k,k'\in[\delta' M]}\frac{1}{N}\sum_{x\in\mathbf{Z}}\Delta_{(2h,2h'),(k,k')}h(x) \geq \frac{\delta^4}{64},$$

which is almost what we want, except that the differencing parameters h and h' are multiplied by 2. This can be rectified with little cost.

Rewriting the left-hand side above as

$$\mathbf{E}_{h,h'\in 2[\delta' M]}\frac{1}{N}\sum_{x\in\mathbf{Z}}\left|\mathbf{E}_{k\in[\delta' M]}\Delta_{(h,h')}h(x+k)\right|^2,$$

we can bound it above by

$$4\mathbf{E}_{h,h'\in[2\delta' M]}\frac{1}{N}\sum_{x\in\mathbf{Z}}\left|\mathbf{E}_{k\in[\delta' M]}\Delta_{(h,h')}h(x+k)\right|^2,$$

extending the average over $2[\delta' M]$ to one over $[2\delta' M]$, which is allowable since every term of the average is positive. Doing the same maneuver with the average over (k, k'), we obtain

$$\mathbf{E}_{h,h',k,k' \in [2\delta' M]} \frac{1}{N} \sum_{x \in \mathbf{Z}} \Delta_{(h,h'),(k,k')} h(x) \geq \frac{\delta^4}{2^{10}}.$$

The conclusion of the lemma now follows upon replacing δ' by $\frac{\delta'}{2}$. □

You will show in the exercises that repeated applications of the Cauchy–Schwarz and van der Corput inequalities can be used more generally to control averages over linear polynomial progressions of any length by a Gowers box norm whose "direction" sets H_i depend only on the leading coefficients.

Exercise 17. Let $N, M, \delta > 0$ and $a_1, \dots, a_m \in \mathbf{Z}$ be nonzero integers. Let $f_0, \dots, f_m : \mathbf{Z} \to \mathbf{C}$ be 1-bounded functions supported on $[N]$. Prove that if

$$\left| \frac{1}{N} \sum_{x \in \mathbf{Z}} \mathbf{E}_{y \in [M]} f_0(x) f_1(x + a_1 x) \cdots f_m(x + a_m x) \right| \geq \delta,$$

then, for all $\delta' \ll_m \delta^{O_m(1)}$, one has

$$\|f_m\|_{\Box^m_{H_0, \dots, H_{m-1}}(N)} \gg_m \delta,$$

where $H_0 = a_m[\delta' M]$ and $H_i = (a_m - a_i)[\delta' M]$ for each $i = 1, \dots, m-1$.

Gowers's proof of Szemerédi's theorem proceeds via a density-increment argument, which begins by using (a variant of) the above statement to show that if a subset $A \subset [N]$ lacks k-term arithmetic progressions, then the U^{k-1}-norm of f_A must be large. The bulk of Gowers's argument is then devoted to proving a (local) *inverse theorem* for U^s-norm for general s, which (roughly, in the case of Gowers's result) classifies 1-bounded functions having large U^s-norm. All proofs of inverse theorems for general U^s-norms are difficult and involved, and far beyond the scope of this course. The proof of the inverse theorem for the U^2-norm, however, is a simple exercise in basic Fourier analysis.

Lemma 7 (U^2-inverse theorem)**.** *Let $N \in \mathbf{N}$, $\delta, \delta' \in (0, 1]$, and $f : \mathbf{Z} \to \mathbf{C}$ be a 1-bounded function supported on the interval $[N]$. If*

$$\|f\|_{U^2_{[\delta' N]}(N)} \geq \delta,$$

then there exists $\xi \in \mathbf{T}$ such that

$$\left| \mathbf{E}_{x \in [N]} f(x) e(\xi x) \right| \geq (\delta \delta')^{O(1)}.$$

Proof. We have

$$\begin{aligned}\delta^4 &\le \frac{1}{N}\sum_{x\in\mathbf{Z}}\mathbf{E}_{h,h',k,k'\in[\delta' N]} f(x+h+k)f(x+h+k')f(x+h'+k)f(x+h'+k') \\ &= \frac{1}{N}\sum_{x\in\mathbf{Z}}\mathbf{E}_{h,h',k,k'\in[\delta' N]} f(x)\overline{f}(x+k'-k)\overline{f}(x+h'-h)f(x+h'-h+k'-k) \\ &= \frac{1}{N}\sum_{x,h,k\in\mathbf{Z}} \mu_{\delta' N}(h)\mu_{\delta' N}(k)f(x)\overline{f}(x+h)\overline{f}(x+k)f(x+h+k)\end{aligned}$$

by making the change of variables $x\mapsto x-h-k$. Applying the Fourier inversion formula for $\mu_{\delta' N}$, this means that

$$\int_{\mathbf{T}^2}\widehat{\mu_{\delta' N}}(\xi)\widehat{\mu_{\delta' N}}(\eta)\frac{1}{N}\sum_{x,h,k\in\mathbf{Z}} f(x)\overline{f}(x+h)\overline{f}(x+k)f(x+h+k)e(\xi h+\eta k)d\xi d\eta \ge \delta^4.$$

Thus, by the triangle inequality,

$$\int_{\mathbf{T}^2}|\widehat{\mu_{\delta' N}}(\xi)\widehat{\mu_{\delta' N}}(\eta)|\left|\frac{1}{N}\sum_{x,h,k\in\mathbf{Z}} f(x)\overline{f}(x+h)\overline{f}(x+k)f(x+h+k)e(\xi h+\eta k)\right|d\xi d\eta \ge \delta^4.$$

and so

$$\left(\int_{\mathbf{T}}|\widehat{\mu_{\delta' N}}(\xi)|d\xi\right)^2\max_{\xi,\eta\in\mathbf{T}}\left|\frac{1}{N}\sum_{x,h,k\in\mathbf{Z}} f(x)\overline{f}(x+h)\overline{f}(x+k)f(x+h+k)e(\xi h+\eta k)\right| \ge \delta^4.$$

The first term is $\frac{1}{(\delta' N)^2}$. Thus, there exist $\xi,\eta\in\mathbf{T}$ such that

$$\left|\frac{1}{N^3}\sum_{x,h,k\in\mathbf{Z}} f(x)\overline{f}(x+h)\overline{f}(x+k)f(x+h+k)e(\xi h+\eta k)\right| \ge (\delta^2\delta')^2.$$

Now, observe that for any finitely supported $g_{00},g_{01},g_{10},g_{11}:\mathbf{Z}\to\mathbf{C}$, we have by the definition of the Fourier transform that

$$\begin{aligned}\int_{\mathbf{T}}\prod_{\omega\in\{0,1\}^2}\mathcal{C}^{|\omega|}\widehat{g}_\omega(\zeta)d\zeta &= \sum_{x_{00},x_{01},x_{10},x_{11}\in\mathbf{Z}}\ \prod_{\omega\in\{0,1\}^2}\mathcal{C}^{|\omega|}g_\omega(x_\omega)\int_{\mathbf{T}} e((x_{00}-x_{01}-x_{10}+x_{11})\zeta)d\zeta \\ &= \sum_{\substack{x_{00},x_{01},x_{10},x_{11}\in\mathbf{Z}\\ x_{00}-x_{01}-x_{10}+x_{11}=0}}\ \prod_{\omega\in\{0,1\}^2}\mathcal{C}^{|\omega|}g_\omega(x_\omega) \\ &= \sum_{x,h,k\in\mathbf{Z}} g_{00}(x)\overline{g_{01}}(x+h)\overline{g_{10}}(x+k)g_{11}(x+h+k),\end{aligned}$$

since the solutions to $x_{00}-x_{01}-x_{10}+x_{11}=0$ are parametrized by $(x,x+h,x+k,x+h+k)$. Thus, setting $f_1(x)=f(x)e(-(\xi+\eta)x)$, $f_2(x)=$

$f(x)e(-\xi x)$, $f_3(x) = f(x)e(-\eta x)$, and $f_4 = f$, we obtain

$$\begin{aligned}(\delta^2\delta')^2 N^3 &\leq \left|\int_{\mathbf{T}} \widehat{f}_1(\zeta)\overline{\widehat{f}_2(\zeta)\widehat{f}_3(\zeta)}\widehat{f}_4(\zeta)d\zeta\right| \\ &\leq \max_\zeta |\widehat{f}_1(\zeta)\widehat{f}_2(\zeta)| \int_{\mathbf{T}} |\overline{\widehat{f}_3(\zeta)}\widehat{f}_4(\zeta)|d\zeta \\ &\leq \|f\|_{\ell^2}^2 \max_\zeta |\widehat{f}_1(\zeta)\widehat{f}_2(\zeta)| \\ &\leq N \max_\zeta |\widehat{f}(\zeta)|^2\end{aligned}$$

by the Cauchy–Schwarz inequality and Parseval's identity. The conclusion of the lemma now follows by dividing through by N and taking the square root. □

7. PET induction and concatenation

Set $M = \sqrt{N}$. Let $f, g, h : \mathbf{Z} \to \mathbf{C}$ be 1-bounded functions supported on the interval $[N]$, and define

$$\Lambda(f, g, h) := \frac{1}{N}\sum_{x\in\mathbf{Z}} \mathbf{E}_{y\in[M]} f(x)g(x+y)h(x+y^2).$$

We will now demonstrate how to control the weighted count of nonlinear Roth configurations by an average of Gowers box norms, following the *PET induction scheme*, which is a method of applying van der Corput's inequality repeatedly to control averages over general polynomial progressions by (averages of) averages over linear progressions. PET induction was a key input in Bergelson and Leibman's proof of the polynomial Szemerédi theorem, and dates back to Bergelson's 1987 "Weakly mixing PET" paper in ergodic theory.

Lemma 8. *Let $f, g, h : \mathbf{Z} \to \mathbf{N}$ be 1-bounded functions supported on the interval $[N]$. If*

$$|\Lambda(f, g, h)| \geq \delta,$$

then, for all $\delta', \delta'' \ll \delta^{O(1)}$, we have

$$\mathbf{E}_{|a|,|b|<\delta' M} \|h\|_{\square^3_{2a[\delta'' M],2b[\delta'' M],2(a+b)[\delta'' M]}(N)} \gg \delta^{O(1)}.$$

Proof. As in the argument controlling $\Lambda_3(f, g, h)$ by a U^2-norm, we first apply the Cauchy–Schwarz inequality followed by van der Corput's inequality to obtain that

$$\frac{1}{N}\sum_{x\in\mathbf{Z}}(1+\delta')\mathbf{E}_a^{\mu_{\delta' M}}\frac{1}{M}\sum_{y\in[M]\cap([M]-a)} g(x+y)\overline{g}(x+y+a)h(x+y^2)\overline{h}(x+(y+a)^2)$$

is at least δ^2. Extending the innermost sum to one over all of $y \in [M]$ introduces an error of size at most $\frac{\delta^2}{2}$ if $\delta' \ll \delta^{O(1)}$, and so by doing this and then making the change of variables $x \mapsto x - y$ we obtain that

$$\frac{1}{N}\sum_{x,a\in\mathbf{Z}} \mu_{\delta' M}(a) g(x)\overline{g}(x+a)\mathbf{E}_{y\in[M]} h(x+y^2-y)\overline{h}(x+(y+a)^2-y)$$

is at least $\frac{\delta^2}{4}$. Applying the Cauchy–Schwarz inequality and using that $\|\mu_{\delta' M}\|_{\ell^2} \leq \frac{1}{\delta' M}$ and $\operatorname{supp}\mu_{\delta' M} \subset (-\delta' M, \delta' M)$, it follows that

$$\mathbf{E}_{|a|<\delta' M}\frac{1}{N}\sum_{x\in\mathbf{Z}} \left|\mathbf{E}_{y\in[M]} h(x+y^2-y)\overline{h}(x+(y+a)^2-y)\right|^2 \geq \frac{\delta^4}{32}.$$

By another application of van der Corput's inequality, we analogously obtain (using that $\delta' \ll \delta^{O(1)}$) that

$$\mathbf{E}_{|a|<\delta' M}\frac{1}{N}\sum_{x,b\in\mathbf{Z}} \mu_{\delta' M}(b)\mathbf{E}_{y\in[M]} h(x+y^2-y)\overline{h}(x+(y+b)^2-(y+b))\overline{h}(x+(y+a)^2-y)h(x+(y+a+b)^2-(y+b))$$

is at least $\frac{\delta^4}{64}$. Making the change of variables $x \mapsto x - y^2 + y$ shows that the above equals

$$\mathbf{E}_{|a|<\delta' M}\frac{1}{N}\sum_{x,b\in\mathbf{Z}} \mu_{\delta' M}(b)\mathbf{E}_{y\in[M]} h(x)\overline{h}(x+2by+b^2-b)\overline{h}(x+2ay+a^2)h(x+2(a+b)y+(a+b)^2-b)$$

which is now an average of averages over a linear polynomial progression. By a final application of the Cauchy–Schwarz inequality, again using that $\|\mu_{\delta' M}\|_{\ell^2} \leq \frac{1}{\delta' M}$ and that $\operatorname{supp}\mu_{\delta' M} \subset (-\delta' M, \delta' M)$, we obtain that

$$\mathbf{E}_{|a|,|b|<\delta' M}\frac{1}{N}\sum_{x\in\mathbf{Z}} \left|\mathbf{E}_{y\in[M]} \overline{h}(x+2by+b^2-b)\overline{h}(x+2ay+a^2)h(x+2(a+b)y+(a+b)^2-b)\right|^2$$

is $\gg \delta^8$. Since the innermost average has size at most 1, there exists a 1-bounded function $f_0 : \mathbf{Z} \to \mathbf{C}$ supported on an interval of length $\ll N$ for which

$$\mathbf{E}_{|a|,|b|<\delta' M}\left|\frac{1}{N}\sum_{x\in\mathbf{Z}} \mathbf{E}_{y\in[M]} f_0(x) f_1^{(a,b)}(x+2by) f_2^{(a,b)}(x+2ay) f_3^{(a,b)}(x+2(a+b)y)\right| \gg \delta^8,$$

where $f_1^{(a,b)}(x) = \overline{h}(x+b^2-b)$, $f_2^{(a,b)}(x) = \overline{h}(x+a^2)$, and $f_3^{(a,b)}(x) = h(x+(a+b)^2-b)$. The conclusion of the lemma now follows from Exercise 17. □

In the next lecture, we will see how to upgrade this to control by a single Gowers uniformity norm.