LECTURE IV STRUCTURE ON $H_*(LM)$

① S^1-ACTION

RECALL $S^1 \times LX \longrightarrow LX$
$(t, \gamma) \longmapsto \gamma(-+t)$.

$\rightsquigarrow H_*(S^1) \otimes H_*(LX) \to H_*(S^1 \times LX) \longrightarrow H_*(LX)$

INTERESTING PART: $[S^1] \in H_1(S^1) \rightsquigarrow H_*(LX) \to H_{*+1}(LX)$.

OR EQUIV. $H^*(LX) \to H^{*-1}(LX)$.

HOCHSCHILD HOMOLOGY: CONNES-RINEHART OPERATOR:

$$B : C_*(A,A) \longrightarrow C_{*+1}(A,A)$$

$$B(a_0 \otimes \cdots \otimes a_k) = \sum_{i=0}^{k} (-1)^{(|a_0|+\cdots+|a_i|)(|a_{i+1}|+\cdots+|a_n|)+in} 1 \otimes a_{i+1} \otimes \cdots \otimes a_n \otimes a_0 \otimes \cdots \otimes a_i$$

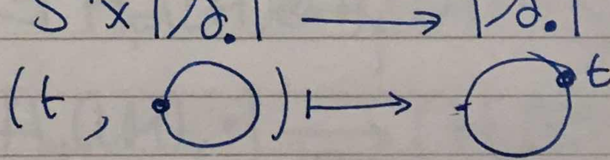
THM (JONES): THESE TWO OPERATIONS CORRESPOND TO EACH OTHER UNDER THE ISOMORPHISM

$$H^*(LX) \cong H_*(C^{-*}(X), C^{-*}(X)).$$

IDEA OF PROOF: $LX = \mathrm{Map}_*(|\mathcal{S}_\bullet|, X)$

S^1-ACTION: $\gamma \in LX \longmapsto [\gamma(-+t) : |\mathcal{S}_\bullet| \xrightarrow{+t} |\mathcal{S}_\bullet| \xrightarrow{\gamma} X]$

NEED TO UNDERSTAND SIMPLICIALLY $S^1 \times |\mathcal{S}_\bullet| \longrightarrow |\mathcal{S}_\bullet|$



$|\mathcal{S}_\bullet| = \coprod_{k \geq 0} \mathcal{S}_k \times \Delta^k / \sim \qquad \mathcal{S}_k = \{ \overset{id}{\bullet\!\!-\!\!\bullet} \cdots \overset{id}{\bullet\!\!-\!\!\bullet},\ \overset{e}{\bullet\!\!-\!\!\bullet}\overset{id}{\bullet\!\!-\!\!\bullet}\overset{id}{\bullet\!\!-\!\!\bullet}, \ldots, \bullet\!\!-\!\!\bullet \cdots \overset{e}{\bullet\!\!-\!\!\bullet} \}$

$\{ \overset{id}{\bullet\!\!-\!\!\bullet} \cdots \overset{id}{\bullet\!\!-\!\!\bullet} \} \times \Delta^k \longmapsto \bullet \in |\mathcal{S}_\bullet| \xrightarrow{+t} t \in |\mathcal{S}_\bullet|$

$\{ \cdots \overset{e}{\bullet\!\!-\!\!\bullet} \cdots \} \times (t_1, \ldots, t_k) \longmapsto t_i \in |\mathcal{S}_\bullet| \longmapsto t_i + t \in |\mathcal{S}_\bullet|$

$$\rightsquigarrow S^1 \times (\Delta_k \times \Delta^k) \longrightarrow \Delta_{k+1} \times \Delta^{k+1}$$

$$(t, t_1 \leq \cdots \leq t_k, \text{"}0\text{"}, \text{"}t_1\text{"}, \ldots, \text{"}t_k\text{"}) \longmapsto (t \leq t_1 + t \leq \cdots \leq t_k + t, \cancel{\text{"}0\text{"}}, \text{"}t\text{"}, \text{"}t_1+t\text{"}, \ldots, \text{"}t_k+t\text{"})$$

CASE 0: $t_k + t \leq 1 \to$ CORRECT ORDERING

CASE 1: $t_{k-1} + t \leq 1 < t_k + t \rightsquigarrow (t_k+t)/\mathbb{Z}$ IN FRONT

⋮

CASE k: $t \leq 1 < t_1 + 1$

$\rightsquigarrow$ SUM OVER $(k+1)$ COPIES OF $(\Delta_{k+1} \setminus \text{"}0\text{"}) \times \Delta^{k+1}$

COROLLARY: HAVE NOW AN EXPLICIT FORMULA TO COMPUTE THE S^1-ACTION ON $H^*(LS^m)$.

EX: OBVIOUSLY 0 ON THE CLASSES $1 \otimes x \otimes \cdots \otimes x$, AND NON-ZERO ON $\underbrace{x \otimes \cdots \otimes x}_{k}$ ODD NBR.

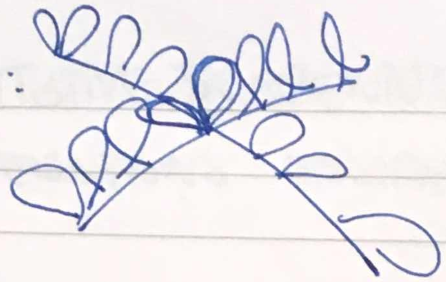
② CHAS-SULLIVAN PRODUCT

CAN INTERSECTION + CONCATENATION OF LOOPS DEFINE A PRODUCT? WHEN M = CLOSED, ORIENT MFD

THM [CHAS-SULLIVAN] $\exists\ H_p(LM) \otimes H_q(LM) \xrightarrow{\bullet} H_{p+q}(LM)$

S.t. $(H_*(M), \underset{\substack{\uparrow \\ \text{INTERSECTION PRODUCT}}}{\bullet}) \xrightarrow{\text{CONST LOOPS}} (H_*(LM), \bullet) \xrightarrow[\text{INTERSEC}]{} (H_*(\Omega M), \underset{\substack{\uparrow \\ \text{CONCAT}}}{*})$

ARE RING MAPS.

IDEA:  GIVEN TWO CHAINS OF LOOPS, INTERSECT THE CHAINS OF BASEPOINTS IN M AND CONCATENATE THE LOOPS AT THE INTERSECTION.

③ ALGEBRAIC MODEL + MORE

ALGEBRAIC MODEL OF $H_*(LM) = HH_*(C^{-*}(M), C^{-*}(M))$

ALGEBRA WITH CUP PRODUCT.

RECALL: INTERSECTION PRODUCT IN $H_*(M)$ $\cong$ CUP PRODUCT IN $H^*(M)$

UNDER POINCARÉ DUALITY.

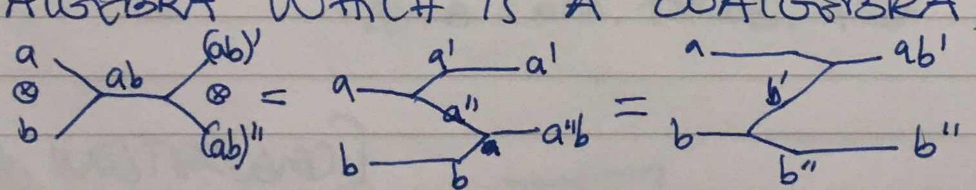
DIGRESSION: FROBENIUS ALGEBRAS.

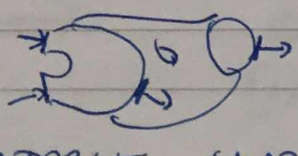
$H^*(M)$ + CUP PRODUCT + PD $\rightsquigarrow$ $H^*(M)$ "POINCARÉ DUALITY ALG" = FROBENIUS ALG

DEF/PROP: A FROBENIUS ALGEBRA

= ALGEBRA WITH A NON-DEGENERATE PAIRING $A \otimes A \xrightarrow{\langle , \rangle} \mathbb{R}$ s.t. $\langle ab, c\rangle = \langle a, bc\rangle$

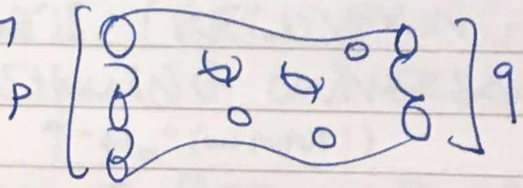
= ALGEBRA WHICH IS A COALGEBRA, S.T.



= OPEN TFT: EVERY TOPOLOGICAL TYPE OF SURFACE WITH p MARKED "INCOMING" AND q "OUTGOING" INTERVALS DEFINES AN OPERATION  $: A^{\otimes p} \to A^{\otimes q}$

AND THESE ARE COMPATIBLE UNDER GLUING.

COMMUTATIVE FROBENIUS ALGEBRAS = CLOSED TFT
= OPERATIONS PARAMETRIZED BY



QUESTION: SUPPOSE A IS A (COMMUTATIVE) FROBENIUS ALGEBRA, WHAT STRUCTURE DOES $C_*(A,A)$ HAVE?

EX: - FOR ANY ALGEBRA A, $C_*(A,A)$ HAS THE B-OPERATOR
$$C_*(A,A) \to C_{*+1}(A,A).$$

- A COMMUTATIVE $\Rightarrow$ $C_*(A,A)$ ADMITS A PRODUCT: THE SHUFFLE PRODUCT + POWER OPERATIONS (MODELS THE CUP PRODUCT ON $H^*(LX)$)

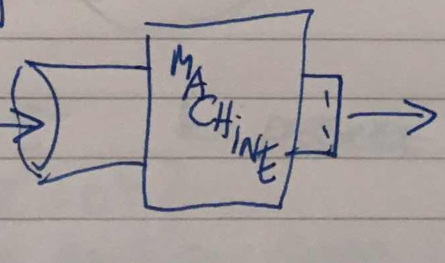
GOOD ENOUGH FOR US?

HAVE $H^*(M)$ IS A COMMUTATIVE FROBENIUS ALG. WHAT ABOUT $C^*(M)$?? SOME HTPY VERSION?...

EASIER: [LAMBRECHTS-STANLEY] $C^*(M) \simeq A_*$ COMM. FROBENIUS (AS A dg-ALG)
WITH $H_*(A_*) \cong H_*(M)$ (AS A COMM. FROB. ALG.) [IN Char 0]

[W-WESTERLAND]

TYPE OF ALGEBRAIC STRUCTURE FOR A
eg: COMM, ASS., A_∞, FROB, POISSON, ...



CHAIN COMPLEX OF NATURAL OPERATIONS ON $C_*(A,A)$ [UNIVERSAL IN SOME SENSE]

INPUT: FROBENIUS ALG $\longrightarrow$ OUTPUT = COMPLEX OF CHORD DIAGRAMS (RECOVERING TRADLER-ZEINALIAN + SHOWING UNIVERSALITY)

↑ "H_0" (ON INPUT!)

INPUT: "OPEN TCFT" = H_*(MODULI OF RIEMANN STRUCT ON SURFACES WITH MARKED INTERVALS IN THEIR BDRY)

⇝ OUTPUT = "CLOSED TCFT"

[RECOVERING COSTELLO + KONTSEVICH-SOIBELMAN + UNIVERSALITY]

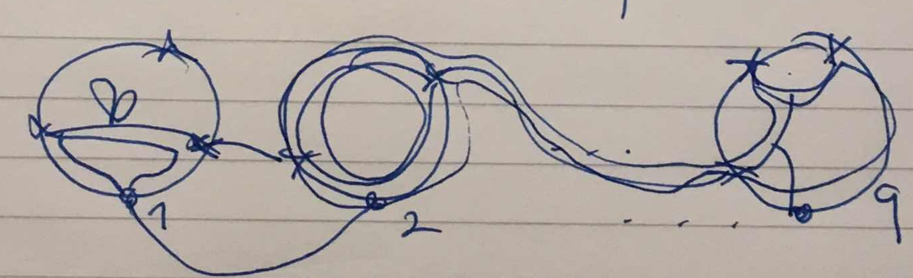
INPUT: COMMUTATIVE FROBENIUS ALGEBRAS

$\longrightarrow$ [ANGELA KLAMT] OUTPUT = "LOOP DIAGRAMS" = STRING TOPOLOGY OPERATIONS.

ROUGHLY: $\forall\, p,q \geq 1$ $\exists$ CHAIN CPX $D_{p,q}$ WITH MAPS

$$D_{p,q} \otimes C_*(A,A)^{\otimes p} \longrightarrow C_*(A,A)^{\otimes q}$$

FOR ANY COMM. FROB ALG A, COMPATIBLE UNDER A CERTAIN COMPOSITION IN $\{D_{p,q}\}$, AND

$D_{p,q} = \mathbb{Z}\Big[$  $\Big]$

- q CIRCLES
- $k \geq 0$ POINTS $\neq 0 \in S^1$ k = DEGREE
- PARTITION OF {POINTS ∪ 0's}
- (WEIGHT ON EACH PARTITION SUBSET)
- p LOOPS BASED AT SPECIAL POINTS

BOUNDARY = COLLIDE POINTS

POSSIBLY CLOSE TO AN EQUIV ON CERTAIN COMP.

[EGAS]: $\exists$ MAPS $C_*(\mathcal{M}_{g,p+q}) \longrightarrow C_*(\overline{\mathcal{M}}^{\#}_{g,p+q}) \xrightarrow{\simeq} D_{p,q}$ + [KATE]