LECTURE III: HOCHSCHILD HOMOLOGY AND A MODEL FOR $H_*(LM)$

LET $\Bbbk$ BE A FIELD. (THINK $\Bbbk = \mathbb{Q}, \mathbb{R}$.)

DEF: A dg-ALGEBRA IS A CHAIN COMPLEX

$$(A = \bigoplus_{n\in\mathbb{Z}} A_n, d) \quad \cdots \leftarrow A_{-1} \xleftarrow{d} A_0 \xleftarrow{d} A_1 \leftarrow \cdots$$
$$d^2 = 0$$

EQUIPPED WITH A MULTIPLICATION

$$m : A_p \otimes A_q \to A_{p+q} \quad \text{CHAIN MAP}$$
$$a \otimes b \mapsto ab$$

WHICH IS ASSOCIATIVE AND UNITAL ($1 \in A_0$).

Ex: • $A = A_0 = \Bbbk$ WITH THE FIELD MULTIPLICATION.

• $A = \Bbbk[x]$ WITH $|x| = 2$, i.e. $x \in A_2$.

$$= A_0 \oplus A_2 \oplus A_4 \oplus \cdots \qquad d = 0.$$
$$= \{ a_0 + \underset{\substack{\cap \\ A_2}}{a_1 x} + \underset{\substack{\cap \\ A_4}}{a_2 x^2} + \cdots + a_\ell x^\ell \mid a_i \in \Bbbk \} \quad \cdots$$

• $A = \Bbbk[x]/x^2 = A_0 \oplus A_2 \qquad d = 0$

• X SPACE $\longrightarrow A = C^{-*}(X)$ = SINGULAR COCHAINS

$$\cdots \leftarrow C^2(X) \xleftarrow{d} C^1(X) \xleftarrow{d} C^0(X)$$

TURNED AROUND SO d GOES DOWN

PRODUCT = CUP PRODUCT.

• X SPACE, $A = H^{-*}(X)$, $d = 0$.

SAME PRODUCT.

<u>NOTE</u>: $X = S^2 \rightsquigarrow A = H^{-*}(S^2) = k[x]/x^2$

$|x| = -2$.

GIVEN A dg-ALGEBRA (A, d, m), DEFINE A NEW CHAIN CPX

$$\left(C_*(A,A) = \bigoplus_{n \geq 0} A \otimes \bar{A}^{\otimes n},\ d = d_A + d_H \right)$$

WHERE
- $k \xrightarrow{1} A \longrightarrow A/1.k =: \bar{A}$
- $|a_0 \otimes \cdots \otimes a_n| = |a_0| + \cdots + |a_n| + n$ (DEGREE)
- $d_A(a_0 \otimes \cdots \otimes a_n) = \sum_{i=0}^{n} (-1)^{|a_0| + \cdots + |a_{i-1}|} a_0 \otimes \cdots \otimes da_i \otimes \cdots \otimes a_n$
- $d_H(a_0 \otimes \cdots \otimes a_n) = \sum_{i=0}^{n-1} (-1)^{i+1} a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n$

$$+ (-1)^{n+1+(|a_n|)(|a_0| + \cdots + |a_{n-1}|)} a_n a_0 \otimes \cdots \otimes a_{n-1}$$

<u>NOTE</u>: SIGNS DEPEND ON CHOICES BUT ARE NECESSARY!

<u>FACTS</u>: 1) $d_A^2 = 0$ (EXERCISES)

2) $d_H^2 = 0$: THIS FOLLOWS FROM $d = \Sigma (-1)^{i+1} d_i$ FOR d_i MAPS SATISFYING THE SIMPLICIAL IDENTITIES.

3) $d_A d_H = d_H d_A$ (i.e. THE d_i's ARE CHAIN MAPS ON $A \otimes \bar{A}^{\otimes n}$)

$$\Rightarrow d^2 = (d_A + d_H)^2 = d_A^2 + d_A d_H + d_H d_A + d_H^2 = 0$$

NEED TO REPLACE d_H BY $(-1)^{|a_0| + \cdots + |a_n|} d_H$

DEF: THE HOCHSCHILD HOMOLOGY OF (A, d, m) IS

$$HH_*(A,A) = H_*(C_*(A,A), d).$$

EX: SUPPOSE A IS AN ALGEBRA $(A = A_0,\ d = 0)$

THEN $$HH_0(A,A) = \frac{\ker(d : C_0(A,A) \to 0)}{\operatorname{Im}(d : C_1(A,A) \to C_0(A,A))} = \frac{A}{[A,A]}$$

$$A \otimes \bar{A} \to A$$
$$a \otimes b \mapsto ab - ba$$

RELEVANCE FOR US:

THM [JONES] FOR X 1-CONNECTED$^{\text{FINITE TYPE}}$ $(\pi_0 X = 0 = \pi_1(X))$, $H_q(X)$ FINITELY GEN. $\forall q$.

$$HH_*(C^{-*}(X), C^{-*}(X)) \simeq H^{-*}(LX)$$

EX: CAN USE THIS TO COMPUTE $H^*(LS^m)$, $m \geq 2$

FACT: S^m IS FORMAL: $C^*(S^m) \simeq H^*(S^m) = k[x]/x^2$

BIG! QUASI-ISOMORPHIC AS A dg-ALG SMALL! $|x| = m$

$$\Rightarrow HH_*(C^{-*}(S^m), C^{-*}(S^m)) \simeq HH_*(H^{-*}(S^m), H^*(S^m))$$

$C_*(H^{-*}(S^m), H^{-*}(S^m))$ GENERATED BY (AS A VECT SP)

$$1,\ x,\ 1 \otimes x,\ x \otimes x,\ 1 \otimes x \otimes x,\ x \otimes x \otimes x,\ \ldots$$

DEG: $0,\ -m,\ 1-m,\ 1-2m,\ 2-2m,\ 2-3m,\ \ldots$

DIFFERENTIAL: $d_A = 0$, $d_H = \sum (-1)^i d_i$

MULTIPLIER TWO ENTRIES $\Rightarrow 0$ MOST OF THE TIME AS $x \cdot x = 0$

EXERCISE: COMPLETE THE COMPUTATION.

(ONLY POSSIBLE NON-ZERO DIFFERENTIALS FROM d_0 AND d_{n}. WHETHER THEY CANCEL OR NOT DEPEND ON n AND m.)

SKETCH PROOF OF JONE'S THEOREM

① $HH_*(C^{-*}(X), C^{-*}(X)) = H_*\left(\bigoplus_{n\geq 0} C^{-*}(X)\otimes \left(\overline{C^{-*}(X)}\right)^{\otimes n}, d = d_A + d_H\right)$

$$\cong H_*\left(\bigoplus_{n\geq 0} (C^{-*}(X))^{\otimes n+1}, d = d_A + d_H\right)$$

$$\cong H_*\left(\bigoplus_{n\geq 0} C^{-*}(X^{n+1}), d = \overline{d_A} + \overline{d_H}\right)$$

ⓐ $AW: C^p(X)\otimes C^q(X) \xrightarrow[\simeq]{\times} C^{p+q}(X\times X)$

$$(\alpha\otimes\beta) \longmapsto \left[\alpha\times\beta : \sigma_{p+q} \mapsto \alpha(\sigma^1|_{\langle v_0,\dots,v_p\rangle})\cdot\beta(\sigma^2|_{\langle v_p,\dots,v_{p+q}\rangle})\right]$$

$$\Delta^{p+q} \xrightarrow{\sigma = (\sigma^1,\sigma^2)} X\times X$$

NOTE: CUP PRODUCT $= C^p(X)\otimes C^q(X) \xrightarrow{AW} C^{p+q}(X\times X) \xrightarrow{D^*} C^{p+q}(X)$

DIAGONAL

$\rightsquigarrow$ $\overline{d_H}$ DEFINED USING DIAGONALS

d_H DEFINED USING CUP PRODUCTS

$\rightsquigarrow$ AW ALMOST GIVES A MAP OF DOUBLE COMPLEXES.

② $X^{\bullet+1}$ IS A COSIMPLICIAL SPACE WITH

$d^i : X^p \longrightarrow X^{p+1}$ GIVEN BY $d^i = X^i\times D\times X^{p-i}$ $i\leq p$

$d_{p+1}(x_1,\dots,x_p) = (x_1,\dots,x_p,x_1)$.

FROM EXERCISES YESTERDAY: $X^{\bullet+1} \cong \mathrm{Maps}(\Delta_\bullet, X)$

AND $|\Delta_\bullet| \cong S^1$.

<u>FACT</u>: $\mathrm{Maps}(|\Delta_\bullet|, X) \simeq \mathrm{tot}(X^{\bullet+1}) = \mathrm{tot}(\mathrm{Maps}(\Delta_\bullet, X))$

"TOTALIZATION" OF THE COSIMPLICIAL SPACE AS IN DESCRIPTION OF LM YESTERDAY.

③ GIVEN A COSIMPLICIAL SPACE $Y^\bullet$ (HERE $X^{\bullet+1}$)

SPACE $\mathrm{tot}(Y^\bullet)$

$\downarrow$

CHAIN CPX: $C^*(\mathrm{tot}(Y^\bullet))$

CHAIN COMPLEX

$\left(\bigoplus_{p\geq 0} C^*(Y^p), d_{\mathrm{sing}} + \sum(-1)^i (d^i)^*\right)$

DOUBLE CPX

$\leadsto$

TOTAL COMPLEX.

COMPUTE THE SAME HOMOLOGY UNDER GOOD CONDITIONS

($\leadsto$ ASSUMPTION X 1-CONN, $\mathbb{k}$ FIELD).

FOR $Y^\bullet = X^{\bullet+1}$, $C^*(\mathrm{tot}(X^{\bullet+1})) = C^*(LX)$

$\left(\bigoplus_{p\geq 0} C^*(X^{p+1}), d_{\mathrm{sing}} + \sum(-1)^i (d^i)^*\right)$

IS THE DOUBLE CPX FROM ①