LECTURE II: SIMPLICIAL SETS AND A MODEL FOR LX

(1) SIMPLICIAL SETS

DEF: A SIMPLICIAL SET $Y_\bullet$ IS A COLLECTION OF SETS Y_k, $k \geq 0$, TOGETHER WITH MAPS

"FACES" $\quad d_i : Y_k \longrightarrow Y_{k-1}$

"DEGENERACIES" $\quad s_i : Y_k \longrightarrow Y_{k+1}$ $\quad \forall\, 0 \leq i \leq k$

SUCH THAT
$$\begin{cases} d_i d_j = d_{j-1} d_i & i<j \\ d_i s_j = s_{j-1} d_i & i<j \\ d_j s_j = id = d_{j+1} s_j \\ s_i s_j = s_{j+1} s_i \end{cases}$$

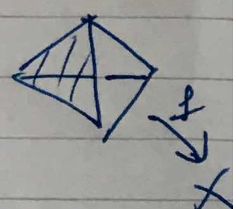
NOTE: IF SOME $Y_k \neq \emptyset$, ALL Y_k'S ARE NON-EMPTY. $\rightsquigarrow$ ∞ TYPE OF OBJECT.

IDEA: • Y_k = SET OF "k-SIMPLICES" (THINK COPIES OF Δ^k)

EX: $Y_k = \text{Maps}(\Delta^k, X) := \text{Sing}_k(X)$ FOR X A SPACE

• EACH d_i RELATES THE FACES OF A k-SIMPLEX WITH ITS FACES AS ELEMENTS OF Y_{k-1}

EX: $d_i : \text{Sing}_k(X) \to \text{Sing}_{k-1}(X)$
$$[f : \Delta^k \to X] \longmapsto [d_i f : \Delta^{k-1} \xhookrightarrow{d^i} \Delta^k \xrightarrow{f} X]$$



• DEGENERACIES s_i FORCE THE EXISTENCE OF DEGENERATE SIMPLICES = IMAGES UNDER SOME s_i — THOSE ARE NEEDED IN MANY CONSTRUCTIONS.

$f \in \mathrm{Sing}_k(X)$

EX: $s_i f : \Delta^{k+1} \xrightarrow{s_i} \Delta^k \xrightarrow{f} X$

" COLLAPSES, BY FORGETTING 1 COORDINATE

→ WHATEVER f WAS, $s_i f$ IS NEVER INJECTIVE.

EXPLICIT DESCRIPTION OF THE d_k^i's AND s^i's USED IN $\mathrm{Sing}_\bullet(X)$:

RECALL: $\Delta^k = \{(t_1,\dots,t_k) \in \mathbb{R}^k \mid 0 \le t_1 \le \dots \le t_k \le 1\}$

$$d^i : \Delta^{k-1} \longrightarrow \Delta^k$$
$$s^i : \Delta^{k+1} \longrightarrow \Delta^k$$

$d^0(t_1,\dots,t_{k-1}) = (0, t_1, \dots, t_{k-1})$

$d^i(t_1,\dots,t_{k-1}) = (t_1,\dots,t_i,t_i,\dots,t_{k-1}) \qquad 0<i<k$

$d^k(\qquad) = (t_1,\dots,t_k,1)$.

$s^i(t_1,\dots,t_{k+1}) = (t_1,\dots,\widehat{t_{i+1}},\dots,t_{k+1})$

↑ FORGET THE $(i+1)$st COORDINATE

EXERCISE: CHECK $\mathrm{Sing}_\bullet(X)$ IS A SIMPLICIAL SET, i.e. THE SIMPLICIAL IDENTITIES ARE SATISFIED.
+ IMPLIES $d^2=0$ FOR $C_*(X)$

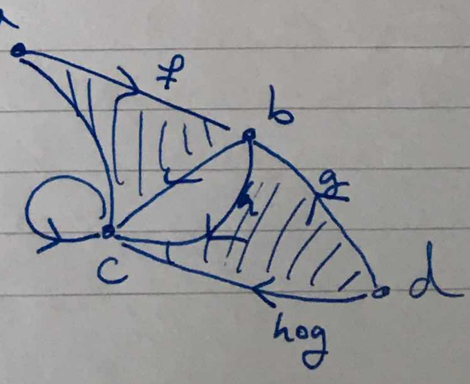
EXAMPLE: $\mathcal{C}$ CATEGORY $\rightsquigarrow$ $N_\bullet\mathcal{C}$ = NERVE OF $\mathcal{C}$
SIMPLICIAL SET.

$N_0\mathcal{C} = \mathrm{Obj}(\mathcal{C})$

$N_1\mathcal{C} = \mathrm{Mor}(\mathcal{C})$

$N_k\mathcal{C} = \{a_0 \xrightarrow{f_1} a_1 \to \dots \xrightarrow{f_k} a_k\}$

COMPOSABLE ARROWS



BOUNDARY MAPS / FACES:

$$d_0(f_1,\dots,f_k) = (f_2,\dots,f_k)$$

$$d_i(f_1,\dots,f_k) = (f_1,\dots,f_{i+1}\circ f_i,\dots,f_k) \qquad 0<i<k$$

$$d_k(\qquad) = (f_1,\dots,f_{k-1})$$

$$s_i(f_1,\dots,f_k) = a_0 \xrightarrow{f_1} a_1 \to \dots \xrightarrow{f_i} a_i \xrightarrow{id_{a_i}} a_i \xrightarrow{f_{i+1}} a_{i+1} \to \dots \xrightarrow{f_k} a_k$$

EXERCISE: CHECK SIMPLICIAL IDENTITIES.

DEF: THE REALIZATION OF A SIMPLICIAL SET $Y_\bullet$ IS THE SPACE

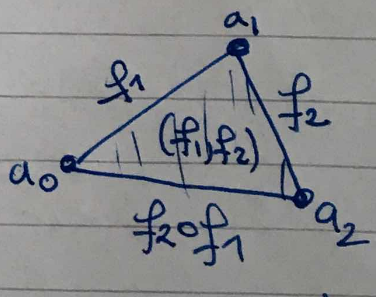
$$|Y_\bullet| = \coprod_{k\geq 0} Y_k \times \Delta^k \Big/ \sim$$

WHERE $(d_i y, \underline{t}) \sim (y, d^i \underline{t})$ $\quad \forall y \in Y_k,\ \underline{t} \in \Delta^{k-1}$

$(s_i y, \underline{t}) \sim (y, s^i \underline{t})$ $\quad \forall y \in Y_k,\ \underline{t} \in \Delta^{k+1}$

→ $|Y_\bullet|$ HAS ONE COPY OF Δ^k FOR EACH k-SIMPLEX σ WITH ITS FACES $\widehat{d_i\sigma}$ ATTACHED TO THE FACES OF THE SIMPLEX, AND DEGENERATE SIMPLICES ARE COLLAPSED IN $|Y_\bullet|$.

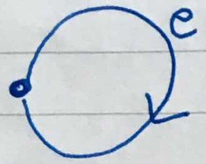
EX: $a_0 \xrightarrow{f_1} a_1 \xrightarrow{f_2} a_2$ in $N_2\mathcal{C}$ $\rightsquigarrow$



in $|N_\bullet\mathcal{C}|$

EXAMPLE: SIMPLICIAL MODEL OF S^1: $\partial_\bullet$

IDEA:



THINK A CATEGORY WITH 1 OBJECT AND 1 NON-IDENTITY MORPHISM e THAT CANNOT BE COMPOSED WITH ITSELF.

$$\partial_0 = \{ \bullet \}$$

$$\partial_1 = \{ \bullet \xrightarrow{e} \bullet \, , \ \bullet \xrightarrow{id} \bullet \}$$

$$\partial_2 = \{ \bullet \xrightarrow{e} \bullet \xrightarrow{id} \bullet \, , \ \bullet \xrightarrow{id} \bullet \xrightarrow{e} \bullet \, , \ \bullet \xrightarrow{id} \bullet \xrightarrow{id} \bullet \}$$

$$\partial_k = \{ \underset{\text{"0"}}{\bullet \xrightarrow{id} \bullet \cdots \bullet \xrightarrow{id} \bullet} \, , \ \underset{\text{"1"}}{\bullet \xrightarrow{e} \bullet \xrightarrow{id} \bullet \cdots \bullet \xrightarrow{id} \bullet} \, , \ \underset{\text{"2"}}{\bullet \xrightarrow{id} \bullet \xrightarrow{e} \bullet \cdots \xrightarrow{id}} \, , \ \underset{\cdots}{\ldots} \, , \ \underset{\text{"k"}}{\bullet \xrightarrow{id} \bullet \cdots \bullet \xrightarrow{id} \bullet \xrightarrow{e} \bullet} \}$$

$\Rightarrow \partial_k$ HAS $k+1$ ELEMENTS.

FACE MAPS: DROP AND COMPOSE AS IN $N_\bullet \mathcal{C}$

DEGENERACIES: INTRODUCE AN id AS IN $N_\bullet \mathcal{C}$.

EX: $|\partial_\bullet| \cong S^1$

② (CO)SIMPLICIAL MODEL FOR LX

$$\begin{aligned}
LX = \mathrm{Map}(S^1, X) &\cong \mathrm{Map}(|\partial_\bullet|, X) \\
&= \mathrm{Map}\Big(\coprod_{k \geq 0} \partial_k \times \Delta^k \big/ \sim \, , X \Big) \\
&\hookrightarrow \mathrm{Map}\Big(\coprod_{k \geq 0} \partial_k \times \Delta^k , X \Big) \\
&= \prod_{k \geq 0} \mathrm{Map}\big(\underbrace{\partial_k \times \Delta^k}_{\coprod_{k+1} \Delta^k} , X \big) \\
&= \prod \mathrm{Map}(\Delta^k, X^{k+1})
\end{aligned}$$

$$F$$

CLAIM: $F : LX \longrightarrow \prod_{k\geq 0} \mathrm{Map}_*(\Delta^k, X^{k+1})$ II (5)

$$\gamma \longmapsto (f_0, f_1, f_2, \dots)$$

$\gamma : S^1 \longrightarrow X$, $\quad S^1 = [0,1]/_{0=1}$ $\qquad f_k : \Delta^k \longrightarrow X^{k+1}$

$$(t_1,\dots,t_k) \longmapsto (\gamma(0), \gamma(t_1), \dots, \gamma(t_k))$$

$$0 \leq t_1 \leq \dots \leq t_k \leq 1$$

PF: THE i^{th} COMPONENT OF f_k IS A MAP $\Delta^k \longrightarrow X$ CORRESPONDING TO THE ith k-SIMPLEX IN $\mathcal{S}_k$

$i=0$ $\quad \overset{id}{\bullet}\!\!-\!\bullet \cdots \overset{id}{\bullet}$ IS THE DEGENERATE SIMPLEX

$$\Delta^k \longrightarrow \Delta^0 \longrightarrow \bullet \in S^1 \xrightarrow{\gamma} X, \qquad \bullet = 0$$

$i>0$ $\quad \overset{id}{\bullet}\!-\!\bullet \cdots \underset{i\text{th position}}{\overset{e}{\bullet\!-\!\bullet}} \cdots \overset{id}{\bullet}$ CORRESPONDS TO $\Delta^k \longrightarrow \Delta^1 \longrightarrow S^1 \xrightarrow{\gamma} X$

$$(t_1,\dots,t_k) \longmapsto t_i$$

$\longrightarrow$ EVALUATED AT t_i.

THM: $LX \cong \{(f_0, f_1, \dots) \in \prod_{k\geq 0} \mathrm{Map}_*(\Delta^k, X^{k+1}) \mid (*)\}$

$(*) = \{$

$f_k(t_1,\dots,t_i,t_i,\dots,t_{k-1}) = D_{i+1} f_{k-1}(t_1,\dots,t_i,\dots,t_{k-1})$

$D_i : X^k \longrightarrow X^{k+1}$ i^{th} DIAGONAL

$f_k(0,t_1,\dots,t_{k-1}) = D_1 f_{k-1}(t_1,\dots,t_{k-1})$

$f_k(t_1,\dots,t_{k-1},1) = D_{0,k+1} f_{k-1}(t_1,\dots,t_{k-1})$

COPIES FIRST TO LAST: $X^k \longrightarrow X^{k+1}$

$U_{i+1} f_{k+1}(t_1,\dots,t_{k+1}) = f_k(t_1,\dots,\hat{t_i},\dots,t_{k+1})$

U_{i+1}: FORGET $(i+1)$st COORD.

PF: THIS IS WHAT THE SIMPLICIAL IDENTITIES DID.

IN FANCY LANGUAGE: $LX = \mathrm{Tot}(\underbrace{\mathrm{Map}_*(\mathcal{S}_\bullet, X)}_{\text{COSIMPLICIAL SPACE}})$

CAN REPLACE $\mathcal{S}_\bullet$ BY ANY SIMPL. SET.