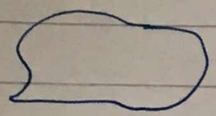
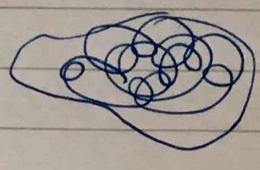
# STRUCTURES ON THE LOOP SPACE

X SPACE $\rightsquigarrow$ $LX = \mathrm{Maps}(S^1, X)$





← ALL CONTINUOUS MAPS
COMPACT-OPEN TOP
$(\{f \mid f(K) \subset U\})$

HUGE SPACE!

WHY STUDY LX?

1) WHY NOT?
2) CAN PRETEND TO DO "PHYSICS"
3) CLOSED GEODESICS ARE THE CRITICAL POINTS OF THE "ENERGY" — ASK NANCY!
4) LX " HAS AN INTERESTING "STRUCTURE" RELATED TO THE MODULI SPACE OF RIEMANN SURFACES. (ASK KATE!)

X MANIFOLD

<u>PLAN</u>: I WHAT CAN YOU DO WITH LOOPS? (PART I)

II SIMPLICIAL MODEL OF LX

$\wr$

III ALGEBRAIC MODEL OF $H_*(LX)$

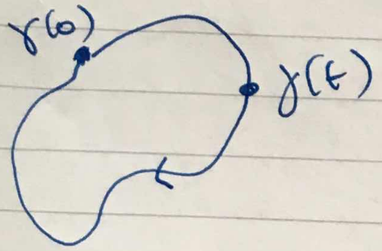
$\wr$

IV WHAT CAN YOU DO WITH LOOPS? (PART II)

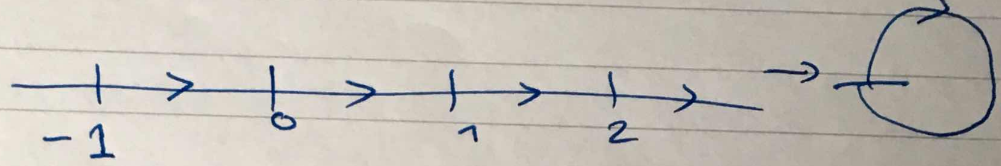
## 1. $S^1$- ACTION : LOOPS CAN BE ROTATED.

FORMALLY : $\exists$ MAP $S^1 \times LX \longrightarrow LX$

$$(t, \gamma) \longmapsto \gamma(t + -)$$




$$S^1 = [0,1]/_{0\sim 1} = \mathbb{R}/\mathbb{Z}$$

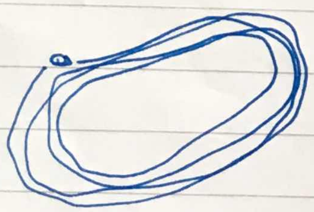



## 2. FLIP : LOOPS CAN BE TURNED AROUND

FORMALLY : $\exists$ MAP $LX \longrightarrow LX$

$$\gamma \longmapsto -\gamma := \gamma((-1).-)$$


## 3. POWER MAPS : CAN RUN A LOOP MANY TIMES.

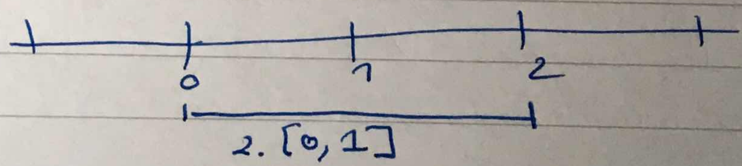


FORMALLY : $\forall k \in \mathbb{Z}$, $\exists$ MAP

$$\theta_k : LM \longrightarrow LM$$

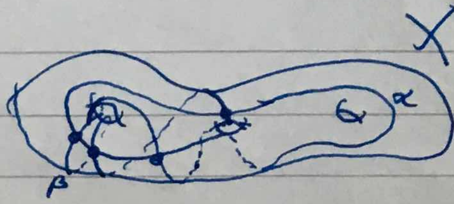
$$\gamma \longmapsto \gamma(k.-)$$

EX: $k=2$



# 4. PRODUCT

REMEMBER: GOLDMAN BRACKET:




$[\alpha, \beta] = \sum \pm \alpha_p \cdot \beta_p$

$\alpha, \beta$ HTPY CLASSES OF LOOPS.

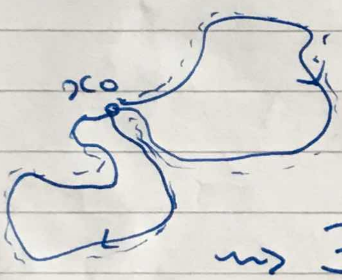
QUESTION: $\exists ?\ LX \times LX \longrightarrow LX$ OF THAT TYPE?

WILL NEED TO GO TO $C_*$ OR $H_*$.

## DIGRESSION 1: BASED LOOP SPACES

$$\Omega X := \mathrm{Map}_*(S^1, X) = \{ f \in LX \mid f(0) = x_0 \}$$

↑ FIXE BASE POINT IN X.




SAME BASEPOINT ⟶ CAN RUN ONE LOOP AFTER THE OTHER

⇝ ∃ CONCATENATION PRODUCT: $\Omega X \times \Omega X \longrightarrow \Omega X$

$$(f, g) \longmapsto f * g$$

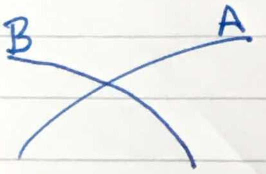
$f * g : [0,1] \longrightarrow X$ RUNS $f$ ON $[0, \frac{1}{2}] \xrightarrow{\cdot 2} [0,1] \xrightarrow{f} X$

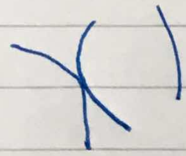
$g$ ON $[\frac{1}{2}, 1] \xrightarrow{\cdot 2 - 1} [0,1] \xrightarrow{g} X$

(USED ABOVE TO DEFINE $\alpha_p \cdot \beta_p$.)

## DIGRESSION 2: THE INTERSECTION PRODUCT.

SUPPOSE $X = M$ IS A MANIFOLD.
SUPPOSE $A, B \subset M$ TRANSVERSE SUBMANIFOLDS




(NOT  )

THEN $A \cap B$ SUBMANIFOLD.

$\dim(A) = p$, $\dim(B) = q$, $\dim(M) = n$

(THINK # EQUATIONS TO DEFINE $A, B, A \cap B$) $\Longrightarrow \dim(A \cap B) = p - (n-q) = q - (n-p)$

$= p + q - n.$

X SPACE

<u>SINGULAR CPX</u>: $C_p(X) = \Bbbk\left[\mathrm{Maps}(\Delta^p, X)\right]$

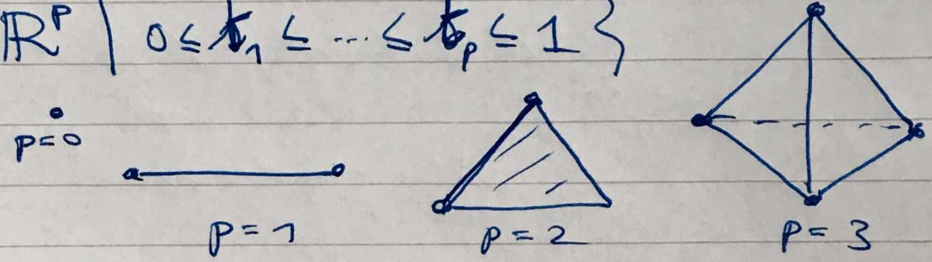
$= \{a_1 f_1 + \dots + a_k f_k \mid a_i \in \Bbbk,\ f_i : \Delta^p \to X\}$

FORMAL SUMS

$\Bbbk = \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \dots$
(RING/FIELD)

$\Delta^p = \{(t_1, \dots, t_p) \in \mathbb{R}^p \mid 0 \le t_1 \le \dots \le t_p \le 1\}$

"p-SIMPLEX"




HAS (p+1) "FACES"

$d^i : \Delta^{p-1} \hookrightarrow \Delta^p \qquad i = 0, \dots, p$

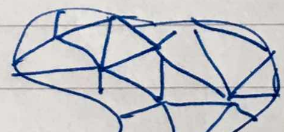
$d : C_p(X) \longrightarrow C_{p-1}(X) \qquad d^2 = 0$ (EX: TOMORROW)

$f \longmapsto \sum_{i=0}^{p} (-1)^i f \circ d^i$ :

$\Delta^{p-1} \xrightarrow{d^i} \Delta^p \xrightarrow{f} X$

$H_*(X) = H_*(C_*(X), d)$.

$A \subset M$ ⇝
dim p, MFD



TRIANGULATE A
→ SUM OF SIMPLICES
$\Delta^p \to M$

⇝ $[A] \in H_p(M)$

<u>THM</u>: ∃ A PRODUCT : $H_p(M) \otimes H_q(M) \to H_{p+q-n}(M)$
(THE "INTERSECTION PRODUCT")
MFD OF dim n

s.t. $A, B \subseteq M$ SUBMANIFOLD, TRANSVERSE

THEN $[A] \cdot [B] = [A \cap B]$.

## DIGRESSION 3: CUP PRODUCT.

$$C^p(X) = \mathrm{Hom}(C_p(X), \Bbbk) \quad \Bbbk = \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \dots$$

$$d: C^p(X) \longrightarrow C^{p+1}(X)$$

$$[\alpha : C_p(X) \to \Bbbk] \longmapsto \Big[ C_{p+1}(X) \xrightarrow[\Sigma(-1)^i d_i]{d} C_p(X) \xrightarrow{\alpha} \Bbbk \Big]$$

$$H^*(X) = H_*(C^*(X), d).$$

CUP PRODUCT: $C^p(X) \otimes C^q(X) \xrightarrow{\cup} C^{p+q}(X)$

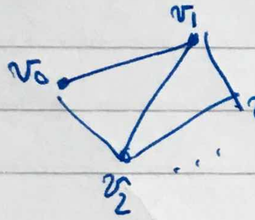
$$(\alpha, \beta) \longmapsto \alpha \cup \beta$$

$$\alpha \cup \beta \, (\sigma_{p+q} = \sigma\langle v_0, \dots, v_{p+q}\rangle) = \alpha(\sigma\langle v_0, \dots, v_p\rangle) \cdot \beta(\sigma\langle v_p, \dots, v_{p+q}\rangle)$$

RESTRICTION OF $\sigma$ TO THE FRONT FACE

RESTRICTION OF $\beta$ TO THE BACK FACE

PRODUCT IN $\Bbbk$




$$\xrightarrow{\sigma_{p+q}} X$$

⇝ INDUCES $H^p(X) \otimes H^q(X) \xrightarrow{\cup} H^{p+q}(X)$

**THM:** IF $M$ IS A MANIFOLD, WE HAVE

$$\begin{array}{ccc} H_{n-p}(M) \otimes H_{n-q}(M) & \xrightarrow[\text{INTERSECTION}]{\bullet} & H_{n-(p+q)}(M) \\ \text{P.D.} \uparrow \cong \cap[M] & & \cong \downarrow \text{PD} \\ H^p(M) \otimes H^q(M) & \xrightarrow{\cup} & H^{p+q}(M) \end{array}$$

$n-(p+q)$

$= (n-p)+(n-q) - n$

PF USES THOM-PONTRJAGIN CONSTRUCTION...