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Amenibility lecture 4

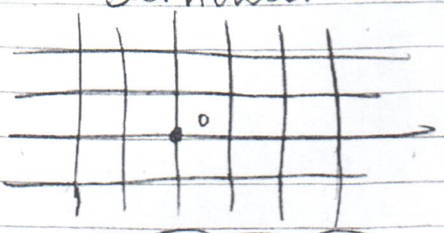
Percolation & amenability

↓

model spreading: Ex's diseases spreading amongan orchard,
fires,
coffee brewing (water & beans)

1957 Broadbent, Hammersely
Bernoulli bond percolation

$T =$



$\forall$ edge $e \in Edges(T)$ toss a $(p, 1-p)$-coin
↳ Bernoulli RV
and declare the edge e open or closed according to the result.

Think of water flowing on the edges

We obtain a Configuration Space:
$\{0,1\}^{Edges}$ equipped with the prob. measure $\mathbb{P}_p = \prod_{e \in Edges} \mu_e$

$$\mu_e(\varphi) = \begin{cases} p & \text{if } \varphi(e) = 1 \\ 1-p & \text{if } \varphi(e) = 0 \end{cases}$$

a configuration

<u>Note</u>: • $(\{0,1\}^{Edges}, \mathbb{P}_p)$ can be viewed as a "random subgraph" of T

• $\mathbb{P}_p$ is G-invariant if $T = Cay(G,S)$

<u>Remark</u> Such a prob space can be similarly defined $(\forall p \in [0,1])$ and $\forall G$ f.g. gp and S a fin. symm^c gen set.

Moreover, it can similarly be defined on any transitive infinite-conn^d graph.

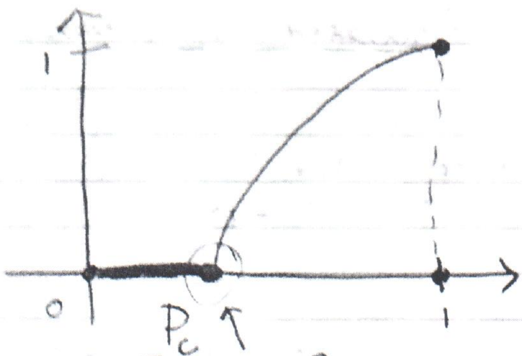
a property that makes a graph similar to $Cay(G, S)$

∃? Given p, are we likely to have an inf conn^d component in our subgraph (an infinite cluster)

$p=0 \Rightarrow$ prob 0 to have an inf. cl.
$p=1 \Rightarrow$ prob 1 to have an inf. cl.

Defn Percolation function

$$\theta(p) = \mathbb{P}_p\{0 \in \text{an infinite cluster}\}$$



$p_c = \sup\{p : \theta(p) = 0\}$

critical value of percolation

cts from left is open for $\mathbb{Z}^3, \dots, \mathbb{Z}^7$
but know for other $\mathbb{Z}^d$

$p_c(\mathbb{Z}^2) = \frac{1}{2}$

$p_c(T_d) = \frac{1}{d-1}$

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Note $p_c = \inf\{p : \overset{\text{almost surely}}{\text{a.s.}}\ \exists \text{ an inf cluster}\}$
$= \sup\{p : \text{a.s. no inf. cluster}\}$

Benjamini, Schramm 1996
"Percolation beyond $\mathbb{Z}^d$; many questions & few answers"

Note $p_c(\mathbb{Z}) = 1$

Conj $p_c(G,S) = 1$ iff G is virtually $\mathbb{Z}$
↳ proved for poly growth gps, exp growth gps, Grigorchik's gps

If $p > p_c$, how many inf clusters do you have?

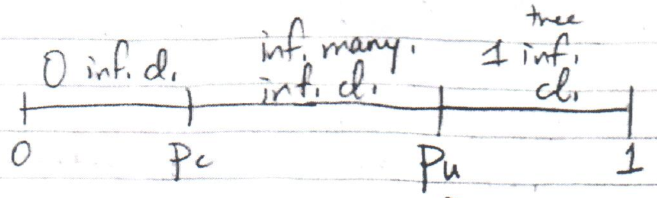
Thm # of inf clusters is almost surely $0, 1, \infty$

Recall Stallings Thm # of ends in a f.g. gp is $0, 1, 2$ or ∞
↑ look up later

Thm In ~~$\mathbb{Z}^2$~~ amenable, $\forall p > p_c$, $\mathbb{P}_p(\exists!\text{ inf cluster}) = 1$

Prop In a tree
$\mathbb{P}_p(\text{the inf cluster is unique}) < 1$
$\forall p < 1$

Grimmet-Neuman $\mathbb{Z} \times T$ tree



unicity parameter

<u>Conjecture</u> (Benjamini, Schramm) (BS-amen. conj)
$p_c = p_u$ iff Γ is amenable

<u>N., Pack ('00)</u> Proved a weak version of the conjecture:
If G is a non-amen f.g. gp then $\exists S \subset G$ a finite gen^g set S (∞ many such) s.t. $\Gamma(G,S)$: $p_c(\Gamma) < p_u(\Gamma)$

Idea of proof combines Spectral Graph Theory & Geometric Group Theory

<u>Corollary</u> A f.g. gp G is amenable iff $\forall S$ fin gen set $p_c(Cay(G,S)) = p_u(Cay(G,S))$

<u>Application.</u> <u>Recall</u> von Neumann's Problem
Does every non-amen G contain a free subgp F_2? No

relax this cond^n

<u>Thm</u> (1999 K. Whyte) G non-amenable $\Leftrightarrow$ its Cayley graph(s) can be partitioned into pieces which are uniformly bi-Lipschitz equiv to T_4.

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Thm (Gaboriau–R. Lyons) 2013
G non-amen. $\Rightarrow$ G admits an action on a prob. space such that almost all orbits of this action can be partitioned into the orbits of an essentially free action of F_2 on this space.

Idea of the proof: (1) G non-amen $\Rightarrow$ weak version of B.S.

$|S| = 2d$ $\exists S$ s.t. in $Cay(G,S) = T$ $p_c < p_u$
Take $p \in (p_c, p_u) \rightsquigarrow (\{0,1\}^{Edges}, \mathbb{P}_p)$
a.s. ∞ many inf. clusters

(2) Use Stalling's Thm: Every conn^d graph with vx degrees $\leq 2d$ can be realized as the Stallings graph of a subgp in F_d.