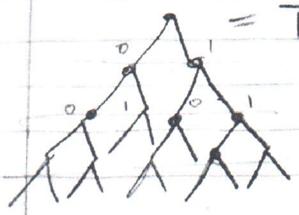
Advanced Lecture 3. 5/24/17

Grigorchuk's gp of intermediate growth
& self-similar gps



$= T = T_2$ (binary tree) consider $\mathrm{Aut}(T)$

$V(T) =$ binary words

$\forall$ aut. preserves distance from root

$\Rightarrow$ action by permutations on each level.

$g \in \mathrm{Aut}(T)$

$$g = \underset{\underset{S_2}{\Uparrow}}{\tau}(g_0, g_1) \qquad g_0 = g|_{T_0} \in \mathrm{Aut}(T) \text{ as } T_0 \equiv T \qquad g_1 = g|_{T(1)}$$

T_0: tree from vx 0

Defn $G < \mathrm{Aut}(T)$ is self-similar if $\forall g \in G, \forall x \in \{0,1\}$
$\exists h \in G,\ y \in \{0,1\}$ s.t. $g(x \cdot w) = y\, h(w)$
w a word $\exists y \in \{0,1\}$

$G = \langle a, b, c, d \rangle$
1st Grigorchuk's gp

$a = \varepsilon(1,1) \qquad \varepsilon \in S_2,\ \neq id$

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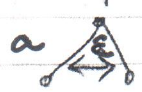
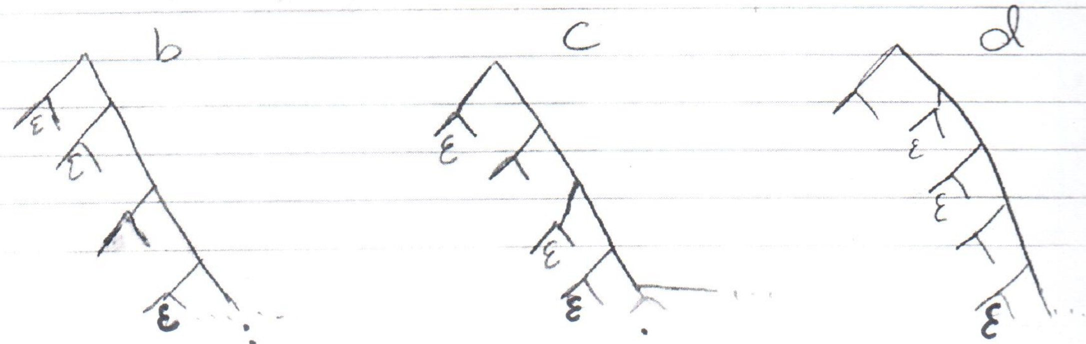


$b = (a, c)$
$c = (a, d)$
$d = (1, b)$



$a^2 = b^2 = c^2 = d^2 = 1$ ... inf. presented gp.
$bcd = 1$

G is a quotient of $\langle a \rangle * \{1, b, c, d\}$
↳ normal forms for elts
$a * a * a * \dots a \qquad * \in \{b, c, d\}$

$G \curvearrowright X \qquad x \in X \quad Stab_G(x) = \{g \in G : gx = x\}$

$v \in V(T) \qquad Stab_G(v)$

$n \in \mathbb{N} \qquad Stab(n) = \bigcap_{|v| = n} Stab(v)$

$|v|$ = length in tree

$G \geq H = Stab(1) = \langle b, c, d, aba, aca, ada \rangle$

<u>Prop 1</u> $\Psi : H \to G \times G$
$g \mapsto (g_0, g_1)$

Ψ is injective & surjective on each factor.

<u>Prop 2</u> The action $G \curvearrowright T$ is contracting i.e.
$\exists C > 0, \quad 0 < \lambda < 1$ s.t.
$|g_i| \leq \lambda |g| + C \qquad i = 0, 1 \quad \forall g \in G$

$|\cdot|$ = length in generators (a, b, c, d).

<u>Cor</u> 1) G is an infinite 2-group, i.e.
$\forall g, \exists k$ s.t. $g^{2^k} = 1$

2) Growth of G is superpolynomial.

More precisely, $\exists \beta > 1/2$ s.t. $\gamma_G(n) \gtrsim e^{n^\beta}$

3) Growth of G is subexponential. More precisely, $\exists \alpha < 1$ s.t. $\gamma_G(n) \lesssim e^{n^\alpha}$.

The fact that $\psi : H \to G \times G$ surj on each level

$\Downarrow$ G is infinite

$\Downarrow$ $\gamma_G \sim \gamma_G^2$

$\Rightarrow \gamma_G(n) \gtrsim \exp^{n^{1/2}}$

The contracting property allows to prove that G is torsion by induction on the length of words.

For upper estimate on the growth, use a stronger version of contracting property:

$\exists\, 0<\mu<1$ $\exists C>0$: $\forall g \in G$ $\exists n$ s.t. $\sum_{|v|=n} |g_v| \leq \mu |g| + C$

Basilica group $B = \langle x, y \rangle < \mathrm{Aut}(T_2)$

$x = (1, y)$ $y = (1, x)\varepsilon$

B is self-similar of exp growth which is not el. amen but amen.

How to check (non)-amenability?

Random Walks $\to$ exactly what you think it is

Polya's Thm Simple Random Walk (SRW) on $\mathbb{Z}$ & $\mathbb{Z}^2$ is recurrent (get back to starting point a.s.)

$\mathbb{Z}^d$ $d \geq 3$ ... transient

Γ conn^d, locally finite graph, assume Γ is d-regular

SRW: $p(u,v)$ = prob to go from u to v in one step

$$= \begin{cases} 1/d & \text{if } u \text{—} v \\ 0 & \text{otherwise} \end{cases}$$

$P = (p(u,v))_{u,v \in V(\Gamma)}$ = transition matrix

$= \frac{1}{d} \mathrm{Adj}(\Gamma)$ adjacency matrix of Γ

$p^{(n)}(u,v)$ = prob going from u to v in n steps

$= \frac{1}{d^n}$ # of paths in Γ between u & v of length n.

Let $(\xi_i)_{i \in \mathbb{N}}$ RW, ξ_n = the position of the RW at time n

Rate of decay of return prob.
= spectral radius of Γ
$= \limsup_{n \to \infty} \sqrt[n]{p^{(n)}(e,e)} = \rho(\Gamma) \leq 1$

<u>H. Kesten (1959)</u> G is amenable $\iff$ $\rho(G,S) = 1$ ($\forall S$ or for some S)

<u>Rate of escape of the RW</u>

$$\ell(\Gamma) = \lim_{n \to \infty} \frac{|\xi_n|}{n} \quad (\exists \text{ limit})$$

<u>Defn</u> Γ is said to have Liouville property if $\ell = 0$.

A function $V(T) \longrightarrow \mathbb{R}$ is harmonic if $Pf = f$.

Constant is harmonic.

Liouville property $\Longleftrightarrow$ no non constant bdd harmonic fth

Thm Liouville property $\Rightarrow$ Amenibility

$\not\Leftarrow$ Counter ex: $\mathbb{Z}_2 \wr \mathbb{Z}^d$ is Liouville iff $d=1,2$

Adian proved that $\rho(B(m,n)) < 1$ if $B(m,n)$ is infinite.

Barholdi-Virag proved that $\ell(\text{Basilica}_{\{x,y\}}) = 0$