Adv Lecture 2

"Amenibility" Tatiana Nagnibeda 5/24/17

Amenibility of G $\iff$ $G \curvearrowright G$ non-paradoxical
$\exists$ of a fin. add. inv. measure

$\Updownarrow$

G satisfies Følner's cond^n
$\hookrightarrow$ $\Gamma = Cay(G,S)$ $\quad i(\Gamma) = 0$

if $i(G,S) > 0$ then $i(G,S)$ depends on S.
But Amenibility is a quasi-isom^c invariant.

Growth $\quad G = \langle S \rangle \quad S = S^{-1}$ finite
$\gamma(n) = \# B_{G,S}(e,n) = \#\{g \in G \mid |g|_S \leq n\}$
growth function.

$\gamma(m+n) \leq \gamma(m) \cdot \gamma(n) \Rightarrow \exists \lim_{n\to\infty} \sqrt[n]{\gamma(n)}$ = rate of exp growth of (G,S) $= \omega(G,S)$

- If $\omega(G,S) = \begin{cases} 1 & G \text{ is of subexp growth} \\ >1 & G \text{ is of exp growth} \end{cases}$

If G is $\mathbb{Z}^d$ $\gamma(n) \sim n^d$

If G is H^3, $\gamma(n) \sim n^4$ Heisenberg gp

Gromov's poly^l growth Thm: If f.g. G is of polynomial growth $\Rightarrow$ G is virtually nilpotent.

Thm Subexp. growth $\Rightarrow$ amenable.

Moreover, You can choose a sequence of balls as Følner sets.

Stability of amenability

The class of amenable gps is closed under:

1) Taking a subgp (Step 3 of pf yesterday)
2) Taking quotients
3) Extensions $1 \to \underset{\substack{\uparrow \\ \text{amen}}}{N} \to G \to \underset{\substack{\uparrow \\ \text{amen}}}{Q} \to 1$ s.e.s. then G is amen

4) Direct limits of groups

Ex of direct limits of infinite gps

$$\varinjlim_{n\to\infty} \mathbb{Z}/p^n\mathbb{Z} = \mathbb{Z}(p^\infty) = \left\{\begin{matrix}\text{roots of unity} \\ \text{of degrees } p^n, n\in\mathbb{N}\end{matrix}\right\}$$

$$S_\infty = \varinjlim_{n\to\infty} S_n$$

These are examples of locally finite gps.

Moreover, we've seen:

$$\begin{matrix}\text{Finite gps} \\ \text{Abelian gps}\end{matrix} \subset \overbrace{\text{Amenable groups}}^{AG}$$

<u>Defn</u> EG = {elementary amenable gps}

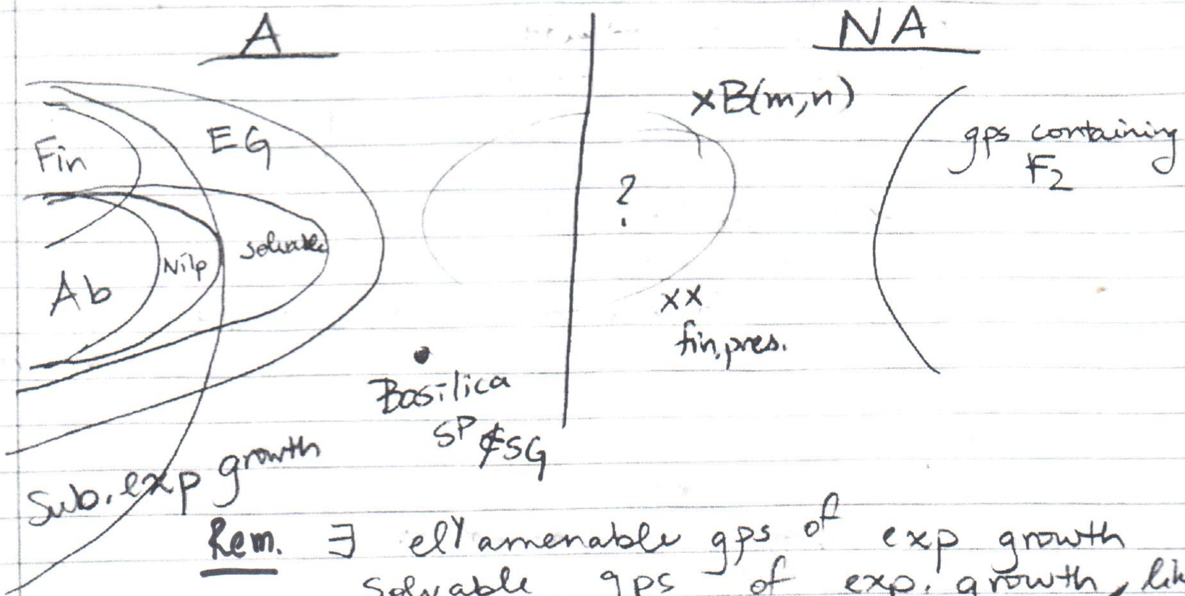
= smallest class of gps containing Fin & Ab & closed under 1) – 4)

Regarding non-ameanble case, we've seen F_2 is non-amenable & $\forall$ gp containing F_2

will also be non-amen (by 1))

Von-Neumann Problem: Does $\forall$ non-amen gp contain F_2? No.

Day's Problem: If $\forall$ amen gp elem amen? No.



Rem. $\exists$ el amenable gps of exp growth
solvable gps of exp. growth, like BS(1,2) or lamplighter $\mathbb{Z} \wr \mathbb{Z}/2\mathbb{Z}$

Chou (1980):
Thm1 If G is an elem. amen. gp & every elt of G has finite order ("torsion" gp or "periodic" gp) then G is locally finite.

Loc fin $\Leftrightarrow$ every f.g. subgp is finite
In. part., a f.g. loc. fin gp is finite!

Cor There is no inf f.g. el am torsion gp.

$$\underset{\text{f.g. ab.}}{G} = \mathbb{Z}^d \oplus \underbrace{F}_{\text{torsion}}$$

<u>Burnside Problem(s) (1902)</u>

Assume G is f.g. gp s.t. $\forall g \in G$ $\exists n$ s.t. $g^n = 1$. (G torsion)
Is G finite?

<u>Bounded version</u>: G f.g. s.t. $\exists n$ s.t. $g^n = e$ $\forall g \in G$ (G is of exponent n)
Is G finite?

- In 1961, infinite f.g. torsion gps
- In 1968, $B(m,n)$ are infinite for $m \geq 2$, $n \geq 1$ odd

free Burnside groups – gen^d by m elts with exponent n.

- In 1980, Grigorchuk construction of a inf f.g. 2-group of intermediate growth

<u>Open</u> Gap in the Growth Conj:

$\exists$? a f.g. gp with

$$Cn^d \leq \gamma(n) \lesssim e^{\sqrt{n}}$$

Grg gp $\quad e^{n^\alpha} \leq \gamma(n) \lesssim e^{n^\beta}$

$$1/2 < \alpha < \beta < 1$$

Recall G has intermediate growth if $\gamma(n)$ is superpolynomial & G is of sub-exp. growth

Chou Thm 2 There is no gp of intermediate growth in EG.

Cor (of Thm 1 or 2) Grigorchuk's example is an amenable but not el. am. gp.

In 1980, Adian: $B(m, n)$ are not amenable if infinite

Thus, inf. $B(m,n)$ are examples of non-amen gps w/o free subgp.

Remark : $\exists$ a f.p. amen not elem amen gp (Grigorchuk)

Questions: 1) Is it possible to replace the "building blocks" = Finite & Ab gps by another class of gps and define its closure under elem ops & show that it is AG?

Try with gps of subexp growth define this class, SG

But $SG \subsetneq AG$ ex: Basilica group

2) Try to formulate & solve "weak" versions of von Neumann problem by relaxing the cond^n "contain F_2".

Positive solutions to geometric & measurable versions → see Lecture 4.

Remarks 1) Chou also proved Thm 3 An inf f.g. elem am gp cannot be simple.

First example of inf f.g. amenable simple gps (not. el. amen.) in 2013: Juschenko-Monod (using work of Matsui & Giordano-Putnam-Skau).

2) Tits Alternative (1972) If G is a f.g. linear gp then G either contains F_2 or is virtually solvable.

3) Open Problem: $\exists$? a f.p. gp of intermediate growth?

Rmk 4) Thompson's gp F (from Susan's talk)

- Not known whether it is amen or not
- It is known that it doesn't contain F_2
- It's known that it is non. elem. amen.

5) Any non-elem. Gromov hyp gp contains F_2.
$G \curvearrowright \partial G$ (use Ping Pong Lemma)