Advanced Course

"Amenability" Tatiana Nagnibeda 5/23/17

History of amenability:

Hausdorff-Banach-Tarski Paradox (1924-1929)

Fun fact about $F_2 = \langle a, b \rangle$

$F_2 > W(a) \sqcup W(a^{-1}) \sqcup W(b) \sqcup W(b^{-1})$

$W(x) = \{\text{reduced words beginning with } x\}$

Consider $a^{-1}W(a) = F_2 \setminus W(a^{-1})$

$\Rightarrow F_2 = aW(a^{-1}) \cup a^{-1}W(a)$ or $F_2 = W(a^{-1}) \sqcup a^{-1}W(a)$

$F_2 = bW(b^{-1}) \cup b^{-1}W(b)$

Remark No HBT Paradox for $\mathbb{R}^1$ or $\mathbb{R}^2$ if you only admit isometries of $\mathbb{R}^1$ or $\mathbb{R}^2$.

Defn $\underset{\text{group}}{G} \curvearrowright \underset{\text{set}}{X}$ The action is paradoxical

(PD) (X is G-paradoxical or admits a paradoxical decomposition wrt G)

if $\exists\ A_1, \dots, A_m, B_1, \dots, B_n$, $n, m \geq 1$
disjoint subsets of X & $\exists$
$g_1, \dots, g_m, h_1, \dots, h_n \in G$ s.t.

$$X = \bigcup_{i=1}^{m} g_i A_i = \bigcup_{j=1}^{n} h_j B_j$$

Equivalent to: $\exists\ A_1, \dots, A_m, B_1, \dots, B_n$, $g_1, \dots, g_m, h_1, \dots, h_n$

(PD') s.t. $X = \bigsqcup g_i A_i = \bigsqcup h_j B_j$

Denote $A = \bigsqcup A_i$ $B = \bigsqcup B_j$

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An equivalence relation on subsets of X
"G-equi-decomposable"

$(PD')\ X \sim_G A \ \& \ X \sim_G B$

Moreover, also equivalent to
strong PD = (SPD) $\exists A, B: \quad X = A \sqcup B \quad X \sim_G A \ \& \ X \sim_G B$

To deduce (SPD) use:

1. (PD') $\underbrace{A \sim_G X \sim_G B \subseteq X \setminus A}_{\precsim}$ & $X \setminus A \precsim X$ trivially

(Cantor-Bernstein-Schröeder)

2. Thm $\forall\ M \sim_G N \iff M \precsim_G N \ \& \ N \precsim_G M$

Thm **HBT Paradox:** $\underbrace{SO(3)}_{\text{gp of rotation in } \mathbb{R}^3} \curvearrowright \overbrace{B^3}^{\text{ball}}$ is paradoxical.

Steps of proof

1. $G \curvearrowright X$, $H \leq G$ and $H \curvearrowright X$ paradoxical $\Rightarrow G \curvearrowright X$ paradoxical.
2. $SO(3) \geq H \cong F_2$ (use rotation by irrational angle) & Ping-Pong Lemma
3. $G \curvearrowright G$ is paradoxical and $G \curvearrowright X$ is free, then $G \curvearrowright X$ is paradoxical.

Indeed, use Axiom of Choice to define $X \supset M$ s.t. M contains one pt from each G-orbit.
!! (Not a measurable set) !!

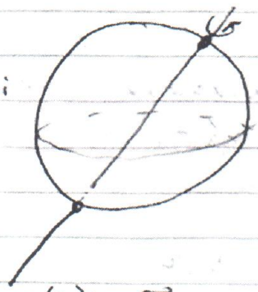
$$X = \bigsqcup_{g \in G} gM = \bigsqcup_{g \in A} gM \sqcup \bigsqcup_{h \in B} hM$$

paradoxical decomp of X

where $G = A \sqcup B$ is paradoxical decomp of G

4. First, instead of $H \curvearrowright B^3$, look at $H \curvearrowright S^{(2)}$
$H \curvearrowright S^{(2)}$ is not free but it only has countablly many fixed pts.

Ex:



2 Pts are fixed.

$H \curvearrowright S^{(2)} \setminus \underbrace{\mathcal{F}}_{\text{countable set of fixed points}}$

Conclusion: This action is paradoxical by the previous step

$\Rightarrow SO(3) \curvearrowright S^2 \setminus \mathcal{F}$ is paradoxical.

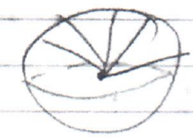
Step 3

5. Lemma, Y_1 G-paradoxical
$Y_2 \sim_G Y_1 \Rightarrow Y_2$ is G-paradoxical

So we prove $S^{(2)} \underset{SO(3)}{\sim} S^{(2)} \setminus \mathcal{F}$

Then $SO(3) \curvearrowright S^2$ is paradoxical

6. $S^2 \to B^3$



Defn (J. von Neuman)
G is amenable if $\exists \mu :$ Subsets$(G) \to [0,1]$
s.t. $\mu(G) = 1$, $\mu(A \sqcup B) = \mu(A) + \mu(B)$,
$\mu(gA) = \mu(A)$ $\forall A < G$, $g \in G$

<u>Ex</u>: G is a finite group then it's amenable!

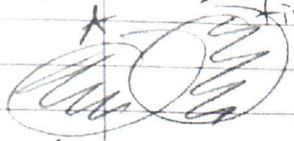
$$\mu(A) = \frac{|A|}{|G|}$$

Having an inv. fin. add. measure is an obstacal to having a paradoxical decomposition.
Indeed, let G be amen.
Assume $G = A \sqcup B$ paradoxical

$$\underset{\mu(A)+\mu(B)}{\overset{}{1 = \mu(G)}} = \mu\Big(\underbrace{\bigsqcup_{i=1}^{n} g_i A_i}_{=1}\Big) + \mu\Big(\underbrace{\bigsqcup_{j=1}^{m} h_j B_j}_{=1}\Big)$$

$$\Rightarrow 1 = 1 + 1 \quad \times$$

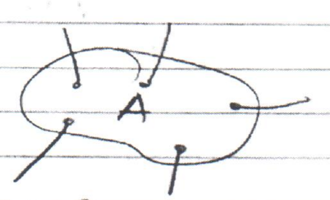
<u>Defn</u> Følner's Condition
G satisfies Følner's condition if $\forall S \subset G$ finite subset, $\forall \varepsilon > 0$ $\exists F \subset G$ a finite subset s.t. $\frac{|sF \triangle F|}{|F|} < \varepsilon \quad \forall s \in S$



shaded = $A \triangle B$

This connects to expanders:
$\Gamma = Cay(G, S) \quad G = \langle S \rangle \quad S = S^{-1}$ finite

$$i(\Gamma) = \inf_{\substack{A \subset G \\ \text{finite}}} \frac{|\partial A|}{|A|}$$



$\partial A = \{$edges between A & $G \setminus A\}$

G satisfies Følner's Cond^n $\Leftrightarrow$ $\Gamma = Cay(G, S)$ has $i(\Gamma) = 0$

How to find "Følner sets" F in G?
$\varepsilon_i \to 0 \rightsquigarrow \{F_i\}$ Følner sets

In the case: $\mathbb{Z}^d$: Balls are Følner sets
Not always true

$\mathbb{Z}$: $[\quad F_n \quad]$ from $-n$ to n; $s \in \{\pm 1\}$
$|sF_n \Delta F_n| = 2 \quad \forall n$
$|F_n| = 2n+1$

Thm: G amen $\overset{(1)}{\Leftrightarrow}$ Følner's Cond^n $\overset{(2)}{\Leftrightarrow}$ $G \curvearrowright G$ non-paradoxical
($\exists$ inv, fin. add meas)

done

$\Leftarrow(1)$ Assume Følner's cond^n
$\{F_n\}$ Følner's sequence for $\varepsilon_n \to 0$

$$\mu_n := \frac{1}{|F_n|} \sum_{x \in F_n} \delta_x$$

δ_x = delta measure on the pt x.

$\mu :=$ an accumulation pt of μ_n's in weak topology
μ will be invariant!

$$|\mu(A) - \mu(sA)| \le |\mu(A) - \mu_n(A)| + \overbrace{|\mu_n(A) - \mu_n(sA)|}^{\text{small by Følner}} + |\mu_n(sA) - \mu(sA)|$$

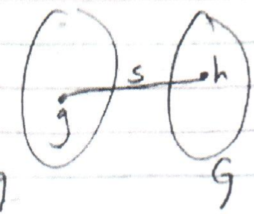
Tarski Thm.
$\Leftarrow(2)$ Assume G violates Følner's Cond^n
$\exists S \subset G$ finite, $\exists \varepsilon > 0$ s.t.
$\forall \underset{\text{finite}}{F \subset G} \quad \exists s \in S$

$$|sF \Delta F| \ge \varepsilon |F|$$

Want to produce a paradoxical decomposition for G.

WLOG we can assume $S \ni e_G$ so $S \cdot F \supset F$
so that $|S \cdot F| \geq (1+\varepsilon)|F|$
By raising to a power k, get $|S^k F| \geq \underbrace{(1+\varepsilon)^k}_{\geq 2}|F|$

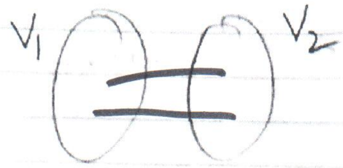
Form a graph:



(g,h) edge $\Leftrightarrow \exists s \in S$ s.t. $h = gs$

$\forall F \subset G, |\mathrm{Neigh}(F)| \geq 2|F|$

Use Hall's Matching Theorem;



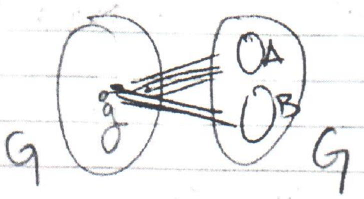
bipartite graph

$\forall X \subset V_1$

$|N(X)| \geq k|X|$

$\Rightarrow \exists k$ matchings that each covers V_1 & the images in V_2 are disjoint.

$\Rightarrow$ We'll have



2 matchings

$A = \bigcup_{s \in S} A_s \quad A_s =$ images of edges labeled by s.

$B = \bigsqcup_{s \in S} B_s$

$\therefore$ paradoxical decomposition for G