Free and Hyperbolic Groups

Olga Kharlampovich

IAS, WAM 2017, Lecture 1

1 Preliminaries

The object of our study - finitely generated groups given by presentations $\langle a_1, ..., a_n \mid r_1, r_2, ...\rangle$, where r_i is a word in $a_1, ..., a_n$. That is groups generated by $a_1, ..., a_n$ with relations $r_1 = 1, r_2 = 1, ...$ imposed.

For example, the free Burnside group of exponent n with two generators is given by the presentation

$$\langle a, b \mid u^n = 1\rangle$$

for all words u in the alphabet a, b.

The fundamental group of the orientable surface of genus n is given by the presentation

$$\langle a_1, b_1, ..., a_n, b_n \mid [a_1, b_1]...[a_n, b_n] = 1\rangle.$$

1.1 Free Group

Let S be an arbitrary set. We define the free group $F(S)$ generated by S, as follows. A *word* w in S is a finite sequence of elements which we write as $w = y_1 \dots y_n$, where $y_i \in S$. The number n is called the *length of the word* w, we denote it by $|w|$. The empty sequence of elements is also allowed. We denote the empty word by e and set its length to be $|e| = 0$.

Consider the set $S^{-1} = \{s^{-1} \mid s \in S\}$ where s^{-1} is just a formal expression. We call s^{-1} the *formal inverse of* s. The set

$$S^{\pm 1} = S \cup S^{-1}$$

is called *the alphabet of* F, and an element $y \in S^{\pm 1}$ of this set is called a *letter*. By s^1 we mean s, for each $s \in S$.

An expression of the type

$$w = s_{i_1}^{\epsilon_1} \dots s_{i_n}^{\epsilon_n} \quad (s_{i_j} \in S;\ \epsilon_j \in \{1, -1\})$$

is called a *group word in* S.

A word w is called **reduced** if it contains no subword of the type ss^{-1} or $s^{-1}s$, for all $s \in S$.

A group G is called a **free group** if there exists a generating set S in G such that every non-empty reduced group word in S defines a non-trivial element of G. If this is the case, then one says that G is *freely generated by* S (or that G is free on S), and S is called a *free basis of* G.

1.2 Construction of a free group with basis S

Let S be an arbitrary set. To construct a free group with basis S, we need to describe a **reduction process** which allows one to obtain a reduced word from an arbitrary word. An *elementary reduction* of a group word w consists of deleting a subword of the type yy^{-1} where $y \in S^{\pm 1}$ from w. For instance, let

$w = uyy^{-1}v$ for some words u and v in S. Then the elementary reduction of w with respect to the given subword yy^{-1} results in the word uv. In this event we write

$$uyy^{-1}v \to uv$$

A *reduction of* w (or a *reduction process starting at* w) consists of consequent applications of elementary reductions starting at w and ending at a reduced word:

$$w \to w_1 \to \cdots \to w_n, \quad (w_n \text{ is reduced})$$

The word w_n is termed a **reduced form of** w.

Proposition 1.1. *Let w be a group word in S. Then any two reductions of w:*

$$w \to w'_0 \to \cdots \to w'_n \quad \textit{and} \tag{1}$$

$$w \to w''_0 \to \cdots \to w''_m \tag{2}$$

result in the same reduced form, in other words, $w'_n = w''_m$.

Exercise 1.2. *Proof the proposition.*

Theorem 1.3. *Let F be a group with a generating set $S \subseteq F$. Then F is freely generated by S if and only if F has the following universal property. Every map $\phi\colon S \to G$ from S into a group G can be extended to a unique homomorphism $\tilde{\phi}\colon F \to G$ so that the diagram below commutes*

$$\begin{array}{ccc} S & \hookrightarrow & F \\ & \phi \searrow & \downarrow \tilde{\phi} \\ & & G \end{array}$$

(here $S \hookrightarrow F$ is the inclusion of S into F).

Exercise 1.4. *Proof the theorem*

If $G = \langle a_1, ..., a_n \mid r_1, r_2, ...\rangle$, then $G = F(A)/ncl(r_1, r_2, \ldots)$.

2 Some classical results for finitely generated groups

The **word problem** is a problem to decide is a given word in a group represents the identity element.

Theorem. (Boone-Novikov's solution of Dehn's problem) There exists a finitely presented group with undecidable word problem.

Theorem. (Higman) A group has recursively enumerable word problem (= the set of words representing the identity) iff it is a subgroup of a finitely presented group.

Theorem. (Adian-Novikov's solution of Burnside problem) The free Burnside group of exponent n with at least two generators is infinite for large enough odd n.

The growth rate of a group is a well-defined notion from asymptotic analysis. To say that a finitely generated group has polynomial growth means the number of elements of length (relative to a symmetric generating set) at most n is bounded above by a polynomial function p(n). The order of growth is then the least degree of any such polynomial function p.

A nilpotent group G is a group with a lower central series terminating in the identity subgroup.

Theorem. (Gromov's solution of Milnor's problem) Any group of polynomial growth has a nilpotent subgroup of finite index.

3 Finitely generated groups viewed as metric spaces

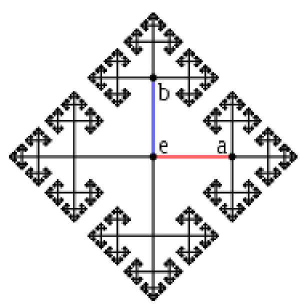
Let G be a group given as a quotient $\pi : F(S) \to G$ of the free group on a set S. Therefore $G = \langle S|R \rangle$. The word length $|g|$ of an element $g \in G$ is the smallest integer n for which there exists a sequence $s_1, \dots, s_n$ of elements in $S \cup S^{-1}$ such that $g = \pi(s_1 \dots s_n)$. The word metric $d_S(g_1, g_2)$ is defined on G by

$$d_S(g_1, g_2) = |g_1^{-1} g_2|.$$

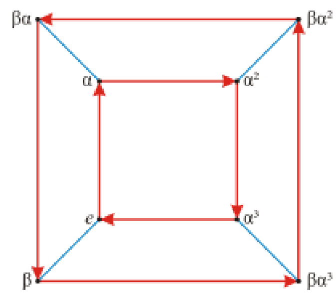
G acts on itself from the left by isometries.

Definition Let $\pi : F(S) \to G$ as above. The corresponding Cayley Graph $Cay(G, S)$ is the graph with vertex set G in which two vertices g_1, g_2 are the ends of an edge if and only if $d_S(g_1, g_2) = 1$.

Cayley graph of the free group $F(a, b)$.



Cayley graph of the Dihedral group $D_4 = \langle \alpha, \beta | \alpha^4, \beta^2, \alpha\beta\alpha\beta \rangle$



3.1 Polynomial growth

A ball of radius n in $Cay(G, S)$ is

$$B_n = \{g \in G || g| \leq n\}.$$

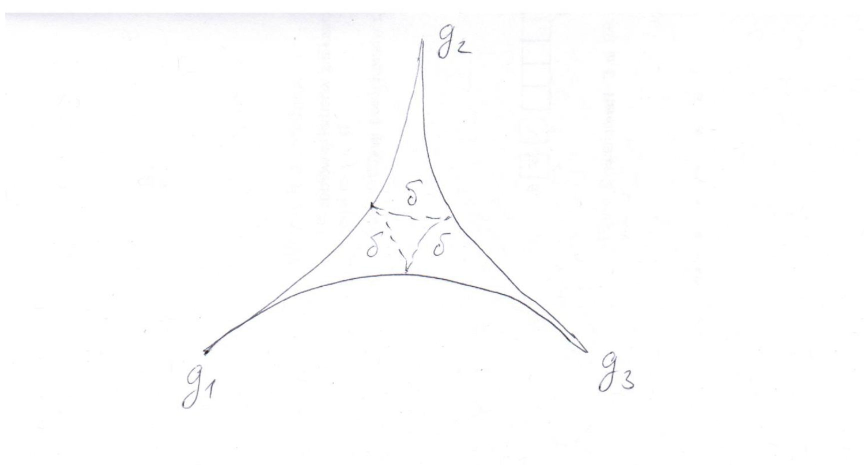
G has polynomial growth iff the number of elements in B_n is bounded by a polynomial $p(n)$.

Exercise 3.1. *What is the growth function of a $F(a, b)$?*

3.2 Hyperbolic groups

A geodesic metric space is called δ-hyperbolic if for every geodesic triangle, each edge is contained in the δ neighborhood of the union of the other two edges. If $\delta = 0$ the space is called a real tree or $\mathbb{R}$-tree.

A group G is **hyperbolic** $Cay(G, X)$ is hyperbolic.



A finitely generated group is called hyperbolic if its Cayley graph is hyperbolic.

4 Quasi-isometry

Definition Let (X, d_X) and (Y, d_Y) be metric spaces. Given real numbers $k \geq 1$ and $C \geq 0$, a map $f : X \to Y$ is called a (k, C)-quasi-isometry if

1. $\frac{1}{k} d_X(x_1, x_2) - C \leq d_Y(f(x_1), f(x_2)) \leq k d_X(x_1, x_2) + C$ for all $x_1, x_2 \in X$,
2. the C neighborhood of $f(X)$ is all of Y.

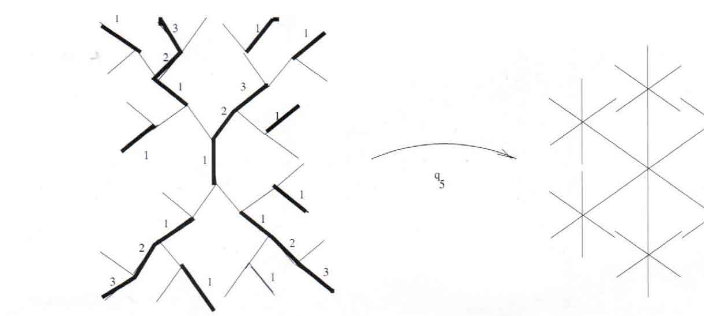
Examples of quasi-isometries

1. $(\mathbb{Z}; d)$ and $(\mathbb{R}; d)$ are quasi-isometric.

The natural embedding of $\mathbb{Z}$ in $\mathbb{R}$ is isometry. It is not surjective, but each point of $\mathbb{R}$ is at most $1/2$ away from $\mathbb{Z}$.

All regular trees of valence at least 3 are quasi-isometric. We denote by T_k the regular tree of valence k and we show that T_3 is quasi-isometric to T_k for every $k \geq 4$. We define the map $q : T_3 \to T_k$, sending all edges drawn in thin lines isometrically onto edges and all paths of length $k - 3$ drawn in thick lines onto one vertex. The map q thus defined is surjective and it satisfies the inequality

$$\frac{1}{k-2} \mathrm{dist}(x, y) - 1 \leq \mathrm{dist}(q(x), q(y)) \leq \mathrm{dist}(x, y).$$



All non-Abelian free groups of finite rank are quasi-isometric to each other. The Cayley graph of the free group of rank n with respect to a set of n generators and their inverses is the regular simplicial tree of valence 2n.

4. Let G be a group with a finite generating set S, and let $Cay(G, S)$ be the corresponding Cayley graph. We can make $Cay(G, X)$ into a metric space by identifying each edge with a unit interval $[0, 1]$ in $\mathbb{R}$ and defining $d(x, y)$ to be the length of the shortest path joining x to y. This coincides with the path-length metric when x and y are vertices. Since every point of $Cay(G, X)$ is in the $1/2$ -neighbourhood of some vertex, we see that G and $Cay(G, S)$ are quasi-isometric for this choice of d.

5. Every bounded metric space is quasi-isometric to a point.

6.

Exercise 4.1. *If S and T are finite generating sets for a group G, then (G, d_S) and (G, d_T) are quasi-isometric.*

7. The main example, which partly justifies the interest in quasi-isometries, is the following. Given M a compact Riemannian manifold, let $\tilde{M}$ be its universal covering and let $\pi_1(M)$ be its fundamental group. The group $\pi_1(M)$ is finitely generated, in fact even finitely presented.

The metric space $\tilde{M}$ with the Riemannian metric is quasi-isometric to $\pi_1(M)$ with some word metric.

8.

Exercise 4.2. *If G_1 is a finite index subgroup of G, then G and G_1 are quasi-isometrically equivalent.*

Corollary 4.3. *If S and $\bar{S}$ are two finite generating sets of G then d_S and $d_{\bar{S}}$ are bi-Lipschitz equivalent.*

Classes of groups complete with respect to quasi-isometries

- Finitely presented groups,
- Nilpotent groups,
- Abelian groups,
- Hyperbolic groups,
- nonabelian free groups of finite rank (follows from the fact that their Caley graphs are trees).
- Amenable groups (Tatiana's lectures)
- Fundamental groups of closed (compact, without boundary) 3-dimensional manifolds. (This is a combination of work of R.Schwartz; M.Kapovich and B.Leeb; A.Eskin, D. Fisher and K.Whyte and, most importantly, the solution of the geometrization conjecture by Perelman.
- Class of fundamental groups of closed n-dimensional hyperbolic manifolds. For $n \geq 3$ this result is due to P. Tukia

4.1 Word problem and conjugacy problem in F

Definition 4.4. (Cyclically reduced word) Let $w = y_1 y_2 \dots y_n$ be a word in the alphabet $S^{\pm 1}$. The word w is cyclically reduced, if w is reduced and $y_n \neq y_1^{-1}$.

Example 4.5. *The word $w = s_1 s_3 s_2^{-1}$ is cyclically reduced, whereas neither $u = s_1 s_2^{-1} s_1 s_3 s_2 s_1^{-1}$, nor $v = s_1 s_3^{-1} s_3 s_2^{-1}$ is a cyclically reduced word.*

Lemma 4.6. *The word and the conjugacy problem in a free group are solvable.*

Observe that there is an (obvious) algorithm to compute both reduced and cyclically reduced forms of a given word w. Our algorithm to solve the word problem is based on

Proposition 4.7. *A word w represents the trivial element in $F(S)$ if and only if the reduced form of w is the empty word.*

Exercise 4.8. *Two cyclically reduced words are conjugate iff one is a cyclic shift of the other.*

4.2 The Isomorphism problem in F

Exercise 4.9. *Let F be freely generated by a set S, and let H be freely generated by a set U. Then $F \cong H$ if and only if $|S| = |U|$.*

4.3 Topological approach

Seifert-van Kampen Theorem Let X be a path-connected topological space. Suppose that $X = U \cup V$ where U and V are path-connected open subsets and $U \cap V$ is also path-connected. For any $x \in U \cap V$, the commutative diagram

$$\begin{array}{ccc} \pi_1(U \cap V, x) & \to & \pi_1(U, x) \\ \downarrow & & \downarrow \\ \pi_1(V, x) & \to & \pi_1(X, x) \end{array}$$

is a push out.

Theorem 4.10. *Let X be a rose with $|S|$ petals - that is, the wedge of $|S|$ copies of S^1 indexed by S. Then $\pi_1(X) = F_S$.*

Proof for finite S: The proof is by induction on $|S|$. If S is empty, we take the wedge of 0 circles to be a point.

Let X be a wedge of $|S|$ circles, let U be (a small open neighbourhood of) the circle corresponding to some fixed element s_0 and let V be the union of the circles corresponding to $T = S \setminus s_0$. $\pi_1(U) = \mathbb{Z}, \pi_1(V) = F_T$ by induction. Let $i : S \to \pi_1(X)$ be the map sending s to a path that goes around the circle corresponding to s.

Consider a set map f from S to a some group G. There is a unique homs $f_1 : \pi_1(U) \to G$ such that $f_1 \circ i(s_0) = f(s_0)$, and unique homs $f_2 : \pi_1(V) \to G$ such that $f_2 \circ i(t) = f(t)$ for all $t \in T$. It follows from the S-van Kampen theorem that there is a unique homs $\hat{f} : \pi_1(X) \to G$ extending f_1 and f_2. QED

This theorem implies that every free group is the fundamental group of a graph (ie a one-dimensional CW complex). This has a strong converse.

Exercise 4.11. *A group is free iff it is the fundamental group of a graph.*

It is enough to show that every graph is homotopy equivalent to a rose. Let Γ be a graph, and let T be a maximal subtree in Γ. Any tree is contractible to a point. Therefore Γ is homotopy equivalent to a rose.

Theorem 4.12. *(Nielsen-Schreier) Every subgroup of a free group is free.*

Proof. Think of a free group F as a fundamental group of a graph X. Let H be a subgroup of F, and let X' be the covering space of X corresponding to H. Then X' is a graph and $H = \pi_1(X')$ so H is free. QED □

Schreier Index Formula (exercise): If H is a subgroup of the free group F_r (rank r) of finite index k, then the rank of H is $1 + k(r-1)$.

4.4 Schreir's graph and Stallings graph

Let $H \leq G$.

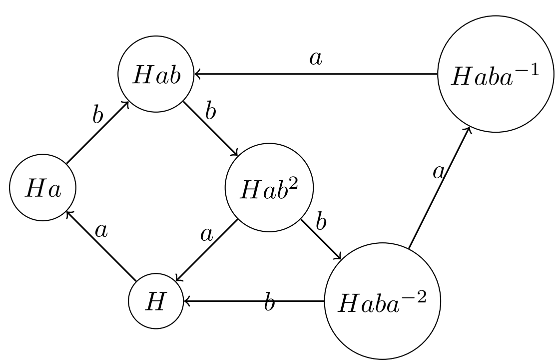
Definition 4.13. The graph of right cosets of H, denoted by $\Gamma_0(H)$, is called the Schreir's graph of H.

$$V(\Gamma_0) = G/H = \{Hg \mid g \in \text{set of right coset representatives}\}$$

$$\forall Hg, \forall s \in S, \exists e \in E_+ \qquad e = (Hg, Hgs) \qquad \lambda(e) = s$$

Definition 4.14. If $G = F(S)$, a free group, $H \leq F$, then the **Stallings graph** $\Gamma(H)$ is the minimal subgraph of $\Gamma_0(H)$, containing all loops at $v_0 = H$.

Example $\Gamma(H)$ for $H = \langle aba^2, a^{-1}b^2, aba^{-2}b\rangle$.



Remark 4.15. *Traversing an edge* $\xrightarrow{a}$ *forward, i.e. along its direction, we read* a*, and traversing it backward, we read* a^{-1}*.*

Definition 4.16. One way reading property (OR) – no two edges outgoing from a vertex are labeled by the same symbol.

Definition 4.17. A path in $\Gamma(H)$ is a sequence $e_1^{\epsilon_1} e_2^{\epsilon_2} \cdots e_n^{\epsilon_n}$, where e_i are edges and $\epsilon_i = \pm 1$. We say that a path is reduced if it contains no subpaths $e_i e_i^{-1}$ or $e_i^{-1} e_i$.

The subgroup H corresponds to labels of loops beginning at v_0 in $\Gamma(H)$:

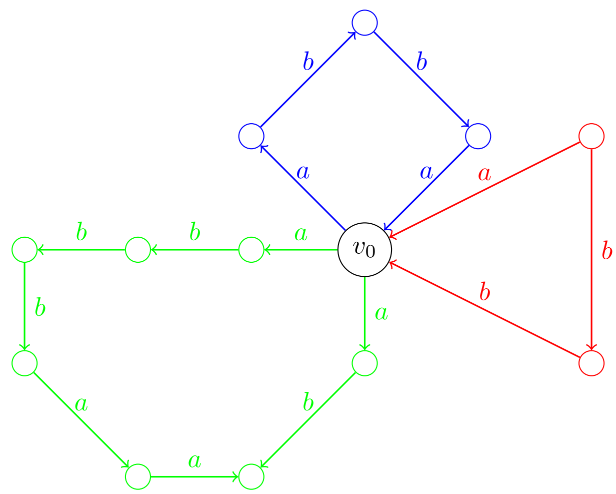
Lemma 4.18. *Every element of H is a loop from v_0 in $\Gamma(H)$.*

This gives an easy procedure to decide whether $g \in H$ of not.
However, how does one contruct $\Gamma(H)$, given $H \leq F$?

4.5 Algorithm to construct $\Gamma(H)$

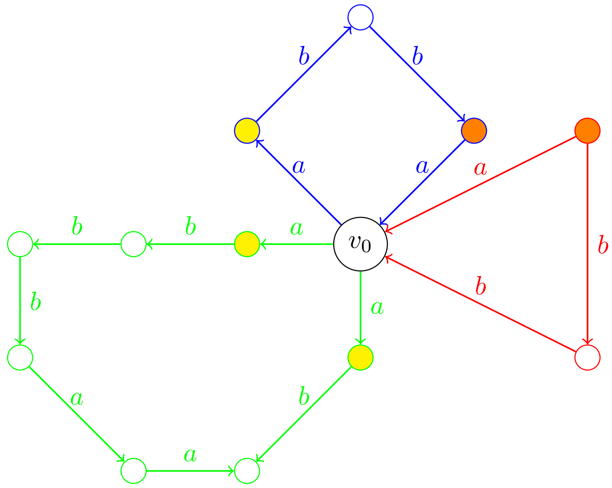
For example, suppose $H = \langle ab^2a, a^{-1}b^2, aba^{-2}b^{-3}a^{-1} \rangle$.

Step 1 Initial graph consists of subdivided circles around a common distinguished vertex $v_0 = H$, each labeled by one of the generators: note that it does not have the OR property!

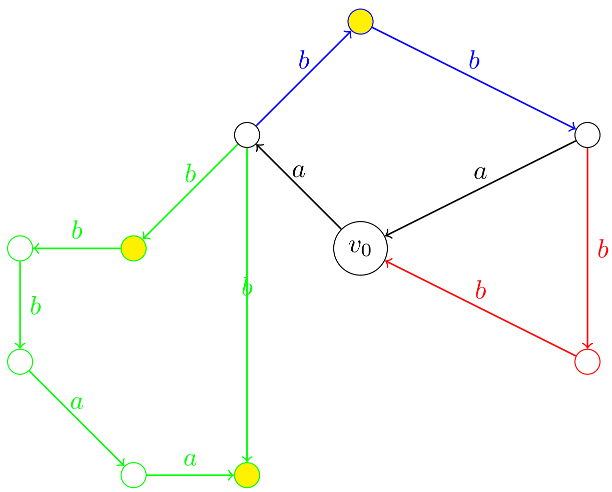


Then, we iteratively identify all edges from the same vertex that have the same label.

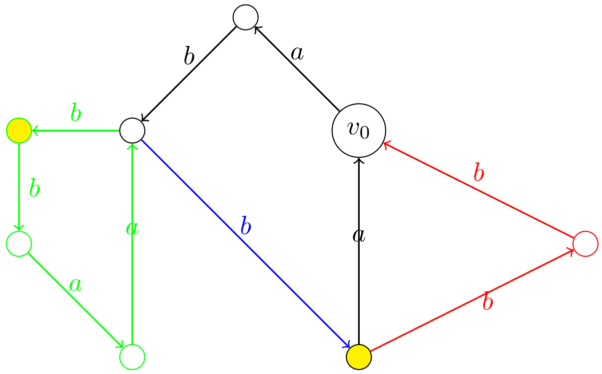
Step 2 Graph before folding. Nodes that are going to be identified are colored.

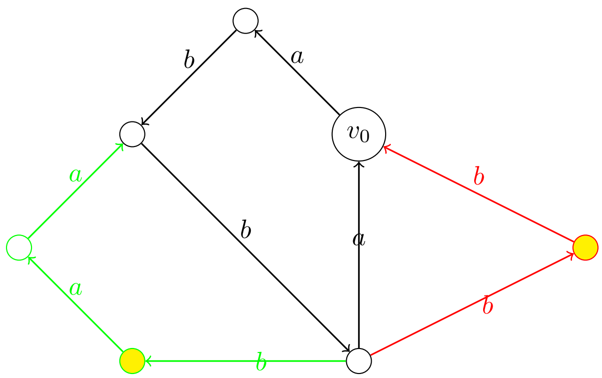


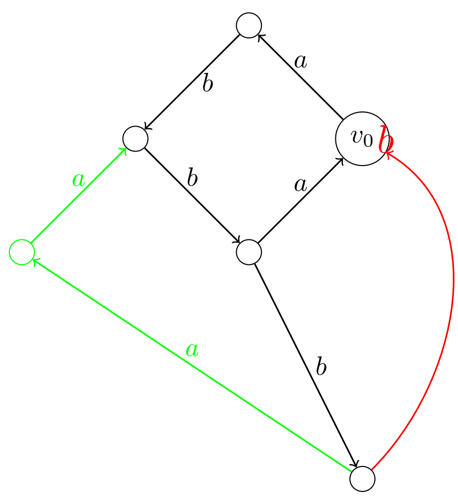
Step 3 Graph after identifying edges labeled by a and a^{-1} coming out of v_0. Nodes to be indetified next are colored again.



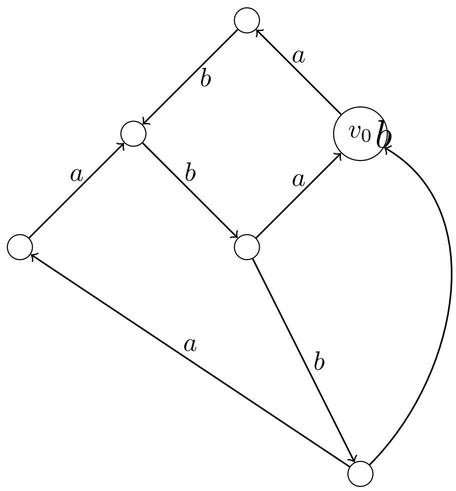
Step 4







Step 5 Final result $\Gamma(H)$, the graph has the OR property.



Exercise 4.19. *Prove that the graph that we obtained is the Stallings graph.*