Linear Algebra Review
Nancy Hingston

A **linear** map $L : \mathbb{R}^n \to \mathbb{R}^m$ is a map of the form

$$L \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} a_{11} \dots a_{1n} \\ \vdots \\ a_{m1} \dots a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

$a_{ij} \in \mathbb{R}$

Example:

$$L : \mathbb{R}^3 \to \mathbb{R}^2 \; L \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x + 2y + 3z \\ 4x + 5y + 6z \end{pmatrix}.$$

An **affine map** $A : \underset{\vec{x}}{\mathbb{R}^m} \to \underset{\vec{y}}{\mathbb{R}^n}$ is a map of the form $A(\vec{x}) = L(\vec{x}) + \vec{y}_0$, $\vec{y}_0 \in \mathbb{R}^n$.

Example:

$$A : \mathbb{R}^3 \to \mathbb{R}^2, A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} 7 \\ 8 \end{pmatrix} = \begin{pmatrix} x + 2y + 3z + 7 \\ 4x + 5y + 6z + 8 \end{pmatrix}.$$

Maximal Rank. We will need to know what affine maps look like in the "nice case", i.e. the case where L has maximal rank. The rank of L is the number of linearly independent rows (or columns) of the matrix (a_{ij}). The maximum possible rank is $min(m, n)$.

a Let $L : \mathbb{R}^n \to \mathbb{R}^n$ be linear. Then L has max rank (i.e. n)
$\Leftrightarrow L$ is injective
$\Leftrightarrow L$ is surjective
$\Leftrightarrow L$ has a continuous inverse
$\Leftrightarrow \; det(a_{ij}) \neq 0$.

b Let $L : \mathbb{R}^n \to \mathbb{R}^{n+k}$ be linear. Then L has max rank (i.e. n)
$\Leftrightarrow L$ is injective
$\Leftrightarrow$ The image of L is an $n-$dimensional plane in $\mathbb{R}^{n+k}$.

c Let $L : \mathbb{R}^{n+k} \to \mathbb{R}^n$ be linear. Then L has max rank (i.e. n)
$\Leftrightarrow L$ is surjective
$\Leftrightarrow$ the preimage $f^{-1}(\vec{y}_0)$ is a $k-$dimensional plane in $\mathbb{R}^{n+k}$ if $\vec{y}_0 \neq 0$.

HINTS for finding the rank of a matrix:

A square matrix has max rank $\Leftrightarrow$ determinant $\neq 0$.

A matrix with 1 row (or column) has max rank (1) unless every entry is 0.

A matrix with 2 rows (or columns) has max rank (2) unless one row (column) is a multiple of the other.

* Inverse of a 2×2 matrix: $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{pmatrix}$.

Facts about **determinants**
$Det : \mathbb{R}^{n^2} \to \mathbb{R}$

1. Det is a group homomorphism from the group of invertible $n \times n$ matrices (with matrix multiplication) to the group of non zero real numbers (with multiplication). Thus

$$Det(A \bullet B) = Det(A) \cdot Det(B)$$

(Note: the multiplication on the left is matrix multiplication. The multiplication on the right is multiplication of real numbers.) And

$$Det(I) = 1$$

where

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \text{identity matrix}$$

2. Det is conjugation invariant, i.e. $Det(A^{-1}BA) = DetB$ (this follows from 1 above; do you see why?)

3. $Det(A) = 0 \Leftrightarrow$ the rows of A are linearly dependent
$\Leftrightarrow$ the columns of A are linearly dependent
$\Leftrightarrow$ the associated linear map $\mathbb{A} : \mathbb{R}^n \to \mathbb{R}^n$ is **not** injective
$\Leftrightarrow$ the associated linear map $\mathbb{A} : \mathbb{R}^n \to \mathbb{R}^n$ is **not** surjective
$\Leftrightarrow$ 0 is an eigenvalue of A

EIGENVALUES
λ is an eigenvalue of $A \Leftrightarrow A\overline{v} = \lambda\overline{v}$ for some $\overline{v} \in \mathbb{R}^n, \overline{v} \neq 0$
$\Leftrightarrow (A - \lambda I)\overline{v} = 0$
$\Leftrightarrow det(A - \lambda I) = 0$

(What does it mean geometrically if $A\overline{v} = \lambda\overline{v}$? What is the significance of the sign?)

Let A be an $n \times n$ matrix.

The characteristic polynomial $P(t) = det(A - tI)$ is a polynomial of degree n. Ex: If $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, then

$$P(t) = det\begin{pmatrix} 1-t & 2 \\ 3 & 4-t \end{pmatrix} = t^2 - 5t - 2.$$

FACT: A always satisfies its characteristic polynomial, i.e. $P(A) = 0$. In above example you can check

$$A^2 - 5A - 2 = 0$$

If $P(t)$ has n distinct (real) roots, then $\mathbb{R}^n$ has a basis of (real) **eigenvectors** for A. That is, we can find a basis $\overline{v}_1, \overline{v}_2, \ldots \overline{v}_n$ for $\mathbb{R}^n$ with

$$A\overline{v}_i = \lambda_i\overline{v}_i$$

for each i, where the eigenvalues λ_i are the roots of the characteristic polynomial. In this case A is **conjugate** to a diagonal matrix, i.e. there is a matrix B (invertible) with $B^{-1}AB = D$ with D diagonal; in fact $D = \begin{pmatrix} \lambda_1 & 0 & & 0 \\ 0 & \lambda_2 & & 0 \\ & & \ddots & \\ 0 & 0 & & \\ 0 & 0 & & \lambda_n \end{pmatrix}$.

The columns of B are the eigenvectors $\overline{v}_1, \ldots, \overline{v}_n$.

Special matrices.

$O(n) =$ Orthogonal Group

$\quad = \{A | A^t A = I\}$

A is **orthogonal** $\Leftrightarrow A^t A = I \Leftrightarrow$ the rows of A form an orthonormal basis for $\mathbb{R}^n$

$\Leftrightarrow$ the columns of A form an orthogobasis for $\mathbb{R}^n$.

$\Leftrightarrow$ The linear map $A : \mathbb{R}^n \to \mathbb{R}^n$ preserves the inner product $< \, , \, >$ on $\mathbb{R}^n$, i.e. $< \overline{x}, \overline{y} > = < A\overline{x}, A\overline{y} > \forall \overline{x}, \overline{y}$

$\Leftrightarrow$ The image of the standard orthonormal basis $\begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \end{pmatrix}, \ldots, \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$ of $\mathbb{R}^n$ is an orthonormal basis for $\mathbb{R}^n$.

Exercise: Show that $O(n)$ is a group, with the operation of matrix multiplication. .

$SO(n)$=Special Orthogonal Group

$= \{A | A^t A = I$ and $\text{Det} A = 1\}$

A is **symmetric** if $A^t = A$.

FACT: If A is symmetric, then all eigenvalues are real and A is **conjugate** by an **orthogonal** matrix Θ to a **diagonal** matrix $D = \begin{pmatrix} \lambda_1 & 0 & & 0 \\ 0 & \lambda_2 & & 0 \\ 0 & 0 & \ddots & \\ 0 & 0 & & \lambda_n \end{pmatrix}$.

That is, $\Theta^{-1} A \Theta = D$.

This means that there is an **orthonormal basis** $\overline{v}_1, \ldots, \overline{v}_n$ for $\mathbb{R}^n$ consisting of eigenvectors for A. The vectors $\overline{v}_1$ are the columns of Θ.

This means that the linear map $\mathbb{A}$ does just what you would expect a diagonal map $\mathbb{D}$ to do.