Beginner's Lecture 4 Kim Ruane

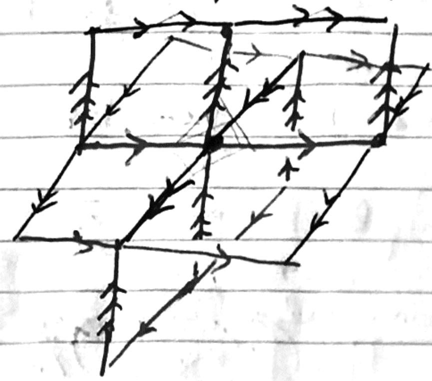
"Examples of Non-Positively curved Groups" 5/26/17

Ex: (from last time) $\langle x, y, z \mid [x,y], z^{-1}xzy^{-1} \rangle$

x

y

z



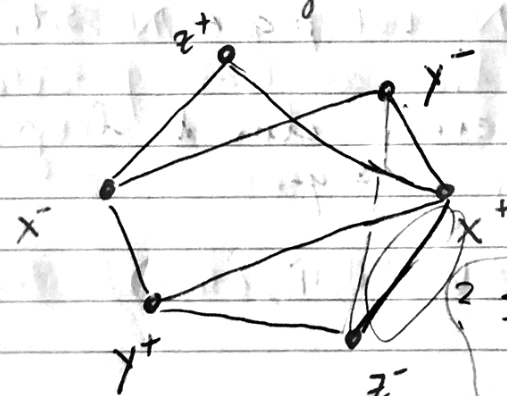
$= (\underset{\langle x \rangle}{\mathbb{Z}} \oplus \underset{\langle y \rangle}{\mathbb{Z}}) * \underset{\langle z \rangle}{\mathbb{Z}}$

Lk(v) - a graph

ε-ball at v

vertices of Lk(v) ⟷ edges at v

edges of Lk(v) ⟷ if the eges bound the same 2-cells



I think the edges have length of the angle?

I think this edge should be in this graph but not drawn on the board

Thm: A simply conn^d 2-complex built from regular Euclidean polygon with all edge lengths 1 is

CAT(0) ⟷ ∀v no circuits of length $< 2\pi$

avoiding pockets of positive curvature at the vertices.

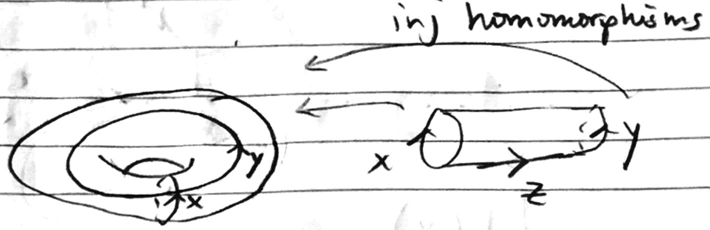
② 5/26/17

$\langle a, b, t \mid tat^{-1} = ab,\ tbt^{-1} = b \rangle$
$= F_2 \rtimes_\rho \mathbb{Z}$ $\rho: a \mapsto ab \quad b \mapsto b$

Tietze's transformations:
$z \leftrightarrow a \qquad x \leftrightarrow t \qquad y \leftrightarrow bt$

* From last example:

HNN extensions $\underset{x}{\mathbb{Z}} \oplus \underset{y}{\mathbb{Z}} *_{\underset{z}{\mathbb{Z}}}$



Let K be a simply conn'd polygonal complex & suppose $\exists N \in \mathbb{N}$ s.t. K contains no n-sided faces for $n > N$. Let $p, q \in \mathbb{N}$ s.t. every face of K has at least p sides and $\forall v \in K$ every closed loop in $Lk(v)$ has length $\geq q$.

The metric is the Euclidean metric w/ each polygon has side length 1.

If $(p,q) \in \{(3,6), (4,4), (6,3)\}$, then K is CAT(0).

pf/ Use regular Euclidean polygons with side length 1.

Thm $A \curvearrowright X_A$, $B \curvearrowright X_B$ are both CAT(0). Then so is $A *_{\mathbb{Z}} B$, $A *_{\mathbb{Z}}$ are too.

5/26/17

Non-Example

"Easy" check for showing a group is NOT CAT(0) is : isoperimetric inequality (must be quadratic)

Rules out: $BS(1,2)$, Heisenberg, $Out(F^n)$

What about: $MCG(S_g)$ $g \geq 3$ has quadratic iso
or $g=2$ 1 bdry

$F_n \rtimes_\varphi \mathbb{Z}$ - Bridson & Groves showed they all have quadratic iso
"very varied"

Non-Ex i.e. not CAT(0)

$F_3 \rtimes_\varphi \mathbb{Z}$ φ: $a \mapsto a$ $c \mapsto ca^2$
$b \mapsto ba$

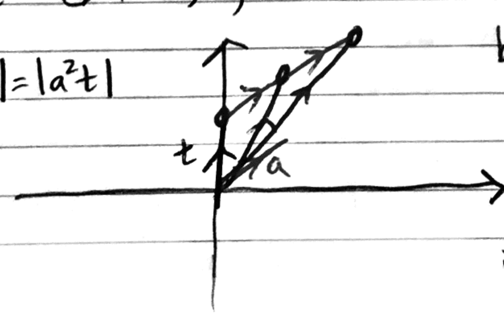
$$\Gamma = \langle a, b, c, t \mid \underbrace{tat^{-1}=a}_{\substack{\mathbb{Z}\oplus\mathbb{Z} \text{ subgp} \\ \langle a,t\rangle}},\ \underbrace{tbt^{-1}=ba}_{b^{-1}tb=at},\ \underbrace{tct^{-1}=ca^2}_{c^{-1}tc=a^2t} \rangle$$

Flat Torus Thm Suppose $\Gamma \curvearrowright X$ geom. and (X is CAT(0))
$A < \Gamma$, $A \cong \mathbb{Z}^n$ there exists an isometrically embedded $\mathbb{R}^n$ on which A acts (geom.) with torus quotient.

If Γ were CAT(0), then

$$|t| = |b^{-1}tb| = |at| = |a^2t|$$

conj of t same translation length



blue & 2 green must all have the same length

This can't happen in $\mathbb{R}^2$; one line cannot intersect a circle in 3 pts

④
5/26/17

$MCG(S_g)$ are not $CAT(0)$

<u>Thm</u> $\Gamma \curvearrowright X \to CAT(0)$ Let $\gamma \in \Gamma$ have infinite order

Then

- Centralizer (γ) [$= C(\gamma)$] acts geom on a convex subset $Y \subseteq X$

  $Y \underset{\text{isometric}}{\equiv} Z \times \mathbb{R}$, Z is convex

- $\exists$ a fin. index subgroup $H \leq C(\gamma)$ s.t

  $H = K \times \langle \gamma \rangle$

for MCG, show that the centralizer of a Dehn twist doesn't satisfy this Thm