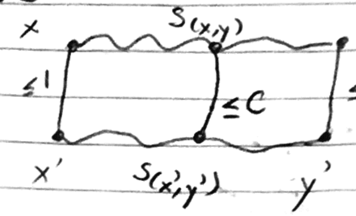
① 5/25/17

Beg Lecture 3 Kim Ruane

Blog: "Here there be dragons" related to Bridson

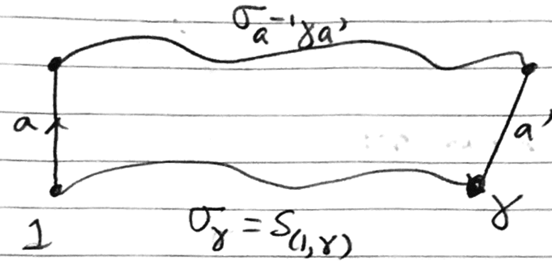
<u>Summary/Lemma</u>: A geodesic m. sp. X is $S^{\text{emi}}H^{\text{yperbolic}}$ iff X admits a cont. q-geod bicombing that satisfies:

$\exists C$ s.t.



$d(s_{x,y}(t), s_{x',y'}(t)) \le C$

$C(\Gamma, A)$



WSH is invariant under QI.

Recall, up to now we've been discussing weak SH.

<u>The extra condition is Equivariance</u>

<u>Defn</u>: If X a m. sp. and X admits an action of Γ (Γ a gp) by isometries then we call X a Γ-space.

<u>Defn</u> X is Γ-semi hyp if X is a Γ-space and X is WSH but the bicombings are Γ-equivariant

$$\gamma \cdot s_{(x,y)} = s_{(\gamma \cdot x, \gamma y)}$$

<u>Defn</u> A f.g. group (Γ, A) is semi-hyp if $X = (\Gamma, A)$ with the Γ-action by left mult is Γ-SH.

<u>Rmk</u> This notion is independent of generating set.

<u>Thm</u> (Bridson/Alonso) - "Svarc-Milner"-ish
Suppose X_1, X_2 are Γ-spaces and $f: X_1 \to X_2$ is a Γ-equivariant Q.I.
If $\Gamma \curvearrowright X_1$ freely and X_2 is Γ-SH then so is X_1. (and conversely - switch the $X_1 \leftrightarrow X_2$ in that statement).

<u>Prop</u> A group Γ is SH does not depend on the gen. set.

pf/ This is because "change of gen set" is bilipshitz.

<u>Note</u> This class is closed under:
- finite index subgroups
- extensions

<u>2 Main Classes</u>: 1) CAT(0) Groups
2) Biautomatic Groups $\in$ SH

<u>Prop</u> CAT(0) Groups are SH.

pf sketch/ $\Gamma \curvearrowright X$ geometrically (prop. discts/cocompact isometric)

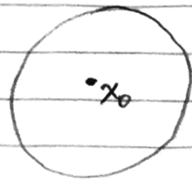
$x_0 \in X$ $\quad$ $\Gamma \to X$, $\gamma \to \gamma \cdot x_0$ is a Γ-equiv Q.I.

(3) 5/25/17

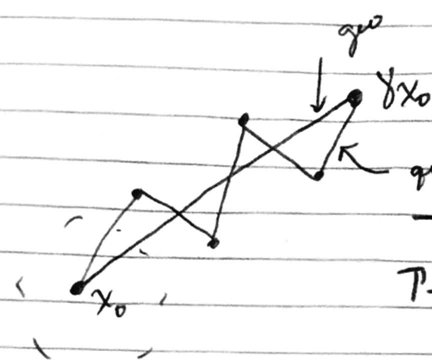
(1) Construct a "nice" gen set for Γ

$\exists D$

$A = \{a \in \Gamma \mid a \cdot D \cap D \neq \emptyset\}$



$\bigcup_{\gamma \in \Gamma} \{\gamma \cdot B(x, D)\}$ cover X



quasi-geo $\Rightarrow$ combing

This provides a Γ-SH structure on X.

A (non-complete) list of Properties that Groups in SH satisfy

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(1) Fin Pres

(2) Quad. Iso. Perimetric

(3) Solvable Conjugacy Problem

$\exists$ a constant $\mu > 0$ s.t. if words $u, v \in F(A)$ represents conjugate elts in Γ iff $\exists w \in F(A)$ of length $\leq \mu^{\max\{|u|,|v|\}}$ s.t. $w^{-1}uw = v$ in Γ

(4) Abelian subgroups are undistorted

i.e. embedding is a quasi-isom embedding

(5) Rich theory of "quasi-convex" subgroups

Centralizers are QC

$\cap$ QC are QC

QC subgroups are SH

(6) No BS behavior; i.e. no $t^{-1}x^p t = x^q$, $|p| \neq |q|$ in SH

CAT(0) Examples

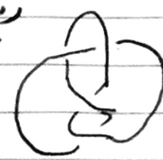
(0) $\Gamma = F_2 \times \mathbb{Z}$, $\Gamma \curvearrowright X = T_4 \times \mathbb{R}$

· Open Question: Are all word hyperbolic groups CAT(0)?

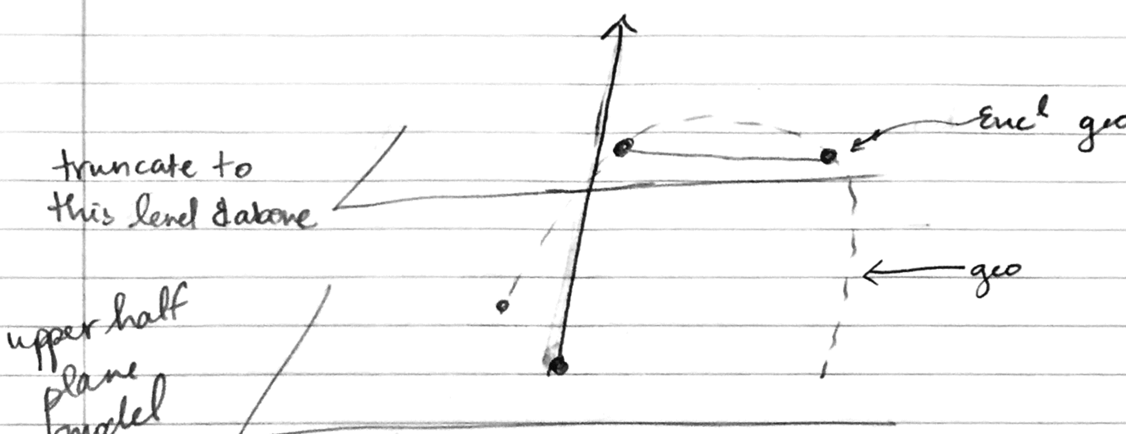
(1) $\underset{\langle a,b\rangle}{F_2} \rtimes_\varphi \mathbb{Z} = \langle a, b, t \mid t^{-1}at = a^2b,\ t^{-1}bt = ab \rangle$

$\mathbb{Z} \to \mathrm{Aut}(F_2)$
$\langle 1 \rangle \mapsto \varphi$

$\pi_1(S^3 / \text{Fig } 8)$
$\downarrow$
acts by isom., properly discts but not cocompactly on $\mathbb{H}^3$



$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \leftarrow \varphi$ defined by
$a \mapsto a^2b$
$b \mapsto ab$



$\mathbb{H}^3 \setminus \bigsqcup \{\text{horoballs}\} = X$

(2) $\Gamma = \langle x, y, t \mid [x,y],\ t^{-1}xt = y \rangle$

$\Lambda = \mathrm{Cay}(\Gamma, A)$
this is CAT(0)