Beg Lecture 2

"Examples of Non-Positively Curved Groups"

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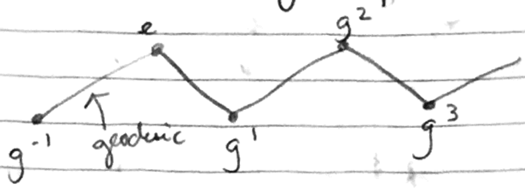
G - word hyperbolic group

$\mathbb{Z}$ subgroups are <u>undistorted</u> in G

$\langle g \rangle$

Consider $\{g^p \mid p \in \mathbb{Z}\}$ in $C(G,S)$



$\exists K, C$ s.t. $\gamma : \mathbb{R} \to C(G,S)$ is a quasi-geodesic line.

We can prove

$$\left| {}^{Cent(g)}\!/_{\langle g \rangle} \right| < \infty \text{ implies } G \text{ contains no } \mathbb{Z} \oplus \mathbb{Z} \text{ subgroups}$$

<u>Summary</u>: G is word hyp $\longleftrightarrow$ G acts geom on a δ hyp space

<u>Non-Defn</u>: G is NPC $\longleftrightarrow$ G acts geom on a "NPC space"

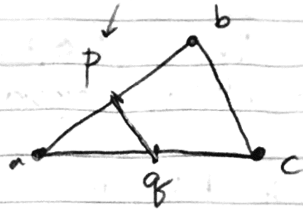
<u>First Real Issue</u>: In $\mathbb{E}^2$ (i.e. should be NPC), quasi geodesics behave badly. They do not satisfy the stability property.

$$\gamma(t) = t(\cos(\ln t), \sin(\ln t)) \quad \text{(Log spiral)}$$

$\gamma : \mathbb{R} \to \mathbb{R}^2$ quasi-geod, but no geod close to it.

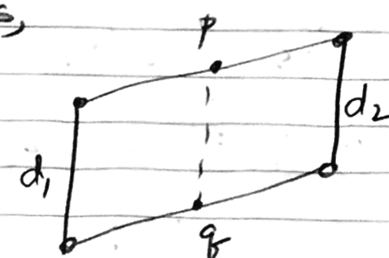
p & q are half-way pts

Property of $\mathbb{E}^2$



$$d(p,q) = \tfrac{1}{2} d(b,c)$$

For parallelograms,



$$d(p,q) = \tfrac{1}{2}(d_1 + d_2)$$

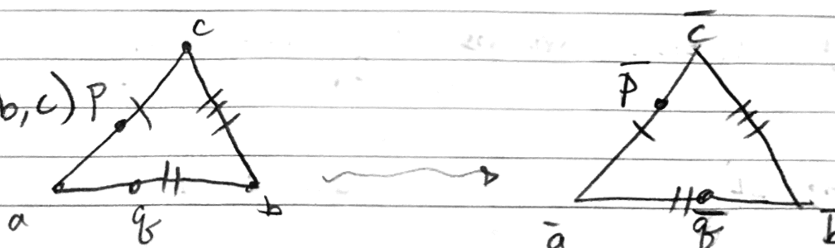
$\mathbb{E}^2$ has curvature 0

CAT(0) inequality

Let X be a proper geod m. sp.
X satisfies the CAT(0) inequality if:

3 names

$\Delta(a,b,c)$



in X in $\mathbb{E}^2$

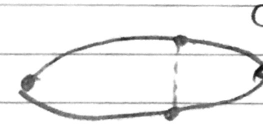
$$d_X(p,q) \le d_{\mathbb{E}^2}(\bar{p},\bar{q})$$

"Δ's are no fatter than flat"

Properties of having CAT(0):

① X has unique geodesics

② X is contractible.
(contract along unique geod)



can't happen

Defn. A group Γ is CAT(0) if Γ acts geometrically on a CAT(0) space X.

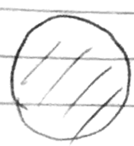
☹ : can't check this in a Cayley graph

↳ trees are CAT(0)

Ex's: $\mathbb{E}^n$, $\mathbb{H}^n$, locally finite metric trees, classical NPC manifold univ cover, Euclidean buildings, lots of 2-dim'l polygonal complexes

☆ Direct Products of these

☆ Gluing constructions that preserve CAT(0)



loop of length n encloses an area $\sim n^2$

$\xrightarrow{\text{Groups}}$ Groups that satisfy a quadratic isoperimetric inequality

All NPC classes groups have this

Ex: $F_3 \rtimes_\varphi \mathbb{Z}$ has quad isoperimetric ineq but is not CAT(0).

Another attempt: Semi-hyperbolic Groups

X – (proper, geodesic) metric space

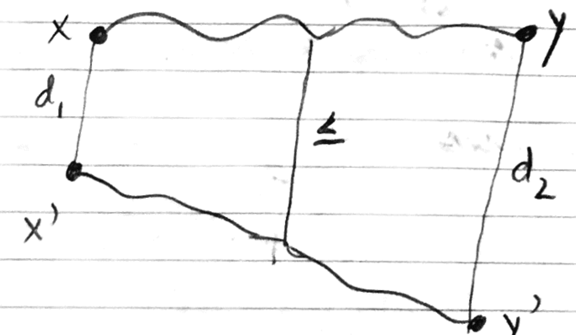
$\mathcal{P}(X) = \{p : \mathbb{N} \to X \mid p \text{ is eventually constant}\}$

Let T_p = largest integer for which $p(T_p) \neq p(T_p - 1)$

Defn: A (discrete) bicombing of X is
$S: X \times X \to \mathcal{P}(X)$
$(x,y) \to S_{(x,y)}$ - combing line btwn x & y

- A bicombing is called q.-geod if $\exists (k,c)$ s.t. $S_{(x,y)}$ is (k,c) quasi-geod $\forall x,y \in X$.
- A bicombing is called bounded if $\exists$ constants $k_1 \geq 1$, $k_2 > 0$ s.t. $\forall x, x', y, y' \in X$, $\forall t \in \mathbb{N}$

combing lines btwn x & y, x' & y'



$$d(S_{(x,y)}(t), S_{(x',y')}(t)) \leq k_1 \max\{d_1, d_2\} + k_2$$

these are really discrete ftns

Defn X is (weakly) semi-hyperbolic if X has a bdd, quasi-geod bicombing.

↓

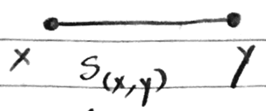
We can check this in a Cayley graph!

Defn A fin gen gp Γ is (weakly) semi-hyp if a Cayley graph is metrically (weakly) semi-hyp.

* This is invariant under Q.I.

Properties: • Γ is f.p. & have quadratic isoperimetric inequality

• δ-hyp m.sps (groups) are semi-hyperbolic



↑ choose from any geodesic

We'll see: CAT(0) gps are semi-hyp