Beginner Lecture
Examples of Non-Positively Curved Groups

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Review of Important ideas & tools

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Cayley Graphs $|S|<\infty$

G a group, $S \subseteq G$, $1 \notin S$, $S = S^{-1}$, $\langle S \rangle = G$

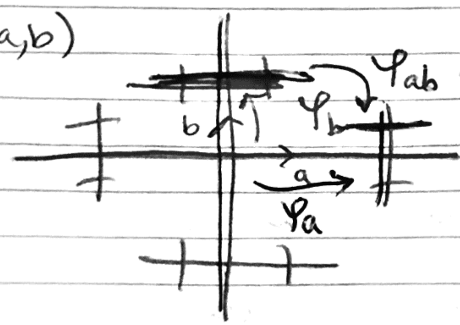
$C(G,S)$ is the Cayley graph of G wrt S

*can do this const^r with any S

$C(G,S)$ is conn^d $\Updownarrow$ S gen^s G

- If we make each edge isometric to $I = [0,1]$ then $C(G,S)$ is a proper, geodesic m sp.
- G acts on $C(G,S)$ by isometries via left multiplication

$F(a,b)$



$\varphi_{ab}(1) = ab$
$\quad\ \ ||$
$\varphi_a \circ \varphi_b(1)$

- The action described is "geometric" - properly discontinuous/cocompact by isometries.

Summary: Every f.g. group acts geometrically on a proper geodesic m. sp.

Svarc-Milner $\rightarrow$ The converse is true.

Exercise (John Meier's Book): $G = S_n$

$S = \{(12), (23), \ldots, (n-1\ n)\}$ i.e. adj trans

$G = \langle S \rangle$

Describe $C(G,S)$

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Cool Property: $C(G,S)$ is bipartite

$\rightarrow$ homom $\varphi: S_n \rightarrow \mathbb{Z}_2$

$\varphi(\sigma) =$ parity of σ (geom)

Quasi-Isometries Revisited

(X, d_X) & (Y, d_Y) are m sp.
$k \geq 1$, $c \geq 0$ Then $f: X \rightarrow Y$ is a (k,c)
quasi-isom embedding:

$$\frac{1}{k} d_X(a,b) - c \leq d_Y(f(a), f(b)) \leq k\, d_X(a,b) + c$$

- If $d_Y(y, f(x)) \leq c$ $\forall y \in Y$ then f is a q. isom.

The 2 most important Examples

(1) For a group G with f.g. sets S_1 & S_2
$C(G, S_1) \sim C(G, S_2)$ (they're quasi-isom^c)

(2) If G acts geometrically on a proper geodesic m. sp. X, then
(fix $x_0 \in X$)

$$g \mapsto g \cdot x_0$$

gives a quasi-isom^y between $G \rightarrow X$.

Some non-examples

(0) bounded space $\not\sim$ unbdd spaces
i.e, finite gps $\not\sim$ inf gps

(1) $[0, \infty) \not\sim \mathbb{R}$

(3)

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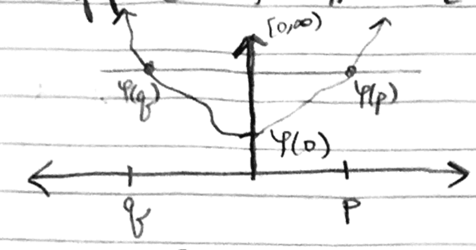
(2) $\mathbb{R}^2 \not\sim \mathbb{R}$

For (1), Suppose $\varphi : \mathbb{R} \to [0,\infty)$ is a q-isom. (k,c) (CONT)



$\lim_{t\to\infty} \varphi(t) = \infty$

$\lim_{t\to-\infty} \varphi(t) = \infty$

q & p are far apart

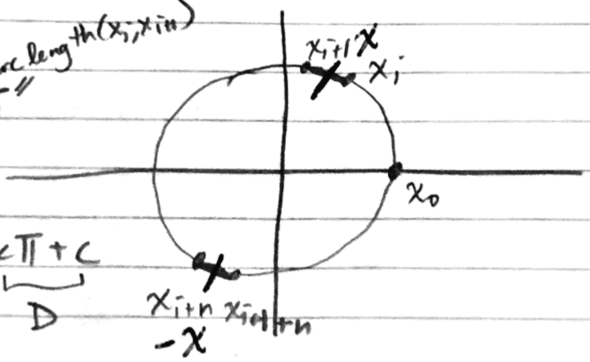
$|\varphi(n) - \varphi(n+1)| \le k+c$

but $\varphi(p)$ & $\varphi(q)$ are within $\sim 2(k+c)$ apart

For (2), Suppose $\varphi : \mathbb{R}^2 \to \mathbb{R}$ a q-isom

Fact: Given any cts ftn $f : S^1 \to \mathbb{R}$ $\exists x \in S^1$ with $f(x) = f(-x)$.

(This can be proved using IVT)



$d(x_i, x_{i+1}) \le \pi$ "arc length(x_i, x_{i+1})"

$d(\varphi(x_i), \varphi(x_{i+1})) \le \underbrace{k\pi + c}_{D}$

Fix $n >$ ______

Take $2n$ equally spaced pts on the circle of radius n centered at $(0,0)$.

Define a cts map $f : S^1 \to \mathbb{R}$ so that $f(x_i) = \varphi(x_i)$.
extend to the arcs between.
So $\exists x$ s.t. $f(x) = f(-x)$.

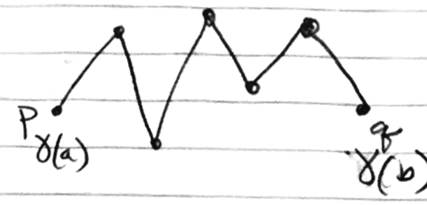
We know $d(x, x_i) < \pi$ & $d(-x, x_{i+n}) < \pi$

$\rightarrow |\varphi(x_i) - \varphi(x_{i+n})| = |f(x_i) - f(x_{i+n})| \le 2D$

We know $\frac{1}{k}|x_i - x_{i+n}| - c \le d(\varphi(x_i), \varphi(x_{i+n}))$

$2\|x_i\| \le k(D+c)$ indep of n

Defn: A (K,C) quasi-geodesic in X is the image of a q-i embedding $\gamma: [a,b] \to X$

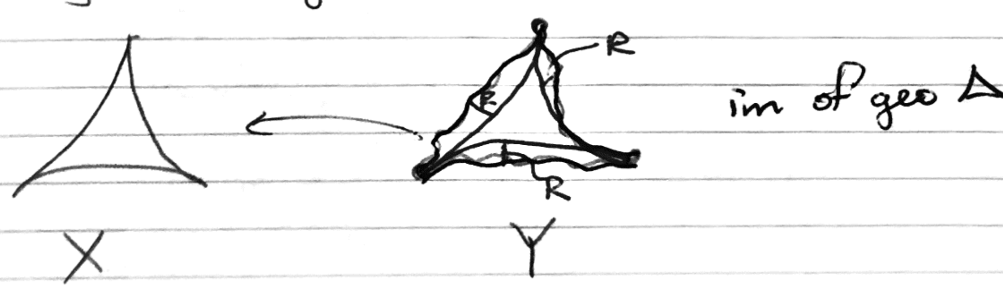


Stability of Quasi-Geodesics in δ-hyp spaces

For all $\delta > 0$, $K \geq 1$, $C \geq 0$, $\exists\, R(\delta, K, C) = R$ s.t. If X is a δ-hyp space & γ is a (K,C) quasi-geod^c in X from p to q with $[p,q]$ any geod^c from p to q in X, then the Hausdorff distance btwn $\mathrm{im}(\gamma)$ & $[p,q] \leq R$.

2 BIG CONSEQUENCES

(1) δ-hyperbolicity is preserved under Q.I. (δ may change)



(2) Allows you to prove that a Q.I. between hyp gps or spaces extends to a homeomorphisms of their boundaries

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i.e. not negatively curved

Ex: Non-positively curved but not hyperbolic

(0) $\mathbb{Z} \oplus \mathbb{Z}$

(1) $F_2 \times \mathbb{Z}$, $F_2 \rtimes_\varphi \mathbb{Z}$