Lectures 1 and 2 Curves and Surfaces in Euclidean space

Images and Preimages

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IAS WAM 2016

Let $f : \mathbb{R}^n \to \mathbb{R}^m$. Here are 3 ways of getting a "picture" of the function f:

(a) f **explicitly** defines its **graph**

$$Graph(f) = \{(\vec{x}, \vec{y}) \in \mathbb{R}^n \times \mathbb{R}^m \mid \vec{y} = f(\vec{x})\} \subseteq \mathbb{R}^{n+m}$$

Example: $f : \mathbb{R}^2 \to \mathbb{R}$; $f\begin{pmatrix} x \\ y \end{pmatrix} = x^2 + y^2$ has as its graph the bowl $z = x^2 + y^2$ (i.e. $\left\{\begin{pmatrix} x \\ y \\ z \end{pmatrix} : z = x^2 + y^2\right\}$) in $\mathbb{R}^3$

Note the graph looks like a copy of the domain in $\mathbb{R}^{n+m}$.

(b) f **parametrically** defines its **image** $f(\mathbb{R}^n) \subseteq \mathbb{R}^m$

Ex: $f : \underset{t}{\mathbb{R}^1} \to \underset{\begin{pmatrix} x \\ y \end{pmatrix}}{\mathbb{R}^2}$; $f(t) = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$ has as its image the circle $\left\{\begin{pmatrix} x \\ y \end{pmatrix} : x^2 + y^2 = 1\right\}$.

Ex: $f : \underset{t}{\mathbb{R}^1} \to \underset{\begin{pmatrix} x \\ y \\ z \end{pmatrix}}{\mathbb{R}^3}$; $f(t) = \begin{pmatrix} \cos t \\ \sin t \\ t \end{pmatrix}$ has as its image the helix $\left\{\begin{pmatrix} x \\ y \\ z \end{pmatrix} : x = \cos z;\ y = \sin z\right\}$

Ex. $f : \underset{\begin{pmatrix} x \\ y \end{pmatrix}}{\mathbb{R}^2} \to \underset{\begin{pmatrix} u \\ v \\ w \end{pmatrix}}{\mathbb{R}^3}$; $f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \\ x^2 + y^2 \end{pmatrix}$

The image of f is the bowl $w = u^2 + v^2$ (i.e. $\left\{\begin{pmatrix} u \\ v \\ w \end{pmatrix} : w = u^2 + v^2\right\}$) in $\mathbb{R}^3$.

Note that the image looks like a copy of the domain sitting in the range. (The image is a subset of the range.)

(c) f **implicitly** defines the **level sets** $f^{-1}(\vec{y}_0) \subseteq \mathbb{R}^n$ (preimages)

Example. $f : \mathbb{R}^2 \to \mathbb{R}^1$; $\quad f\begin{pmatrix} x \\ y \end{pmatrix} = x^2 + y^2$. $\quad$ If $c \in \mathbb{R}^1$,

$$f^{-1}(c) = \begin{cases} \text{a circle if } c > 0 \\ \text{a point if } c = 0 \\ \varnothing \text{ if } c < 0. \end{cases}$$

We can sketch the level sets in the domain $\mathbb{R}^2$:

Ex $f : \mathbb{R}^3 \to \mathbb{R}^1$; $\quad f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = x^2 + y^2$. $\quad$ If $c \in \mathbb{R}^1$,

$$f^{-1}(c) = \begin{cases} \text{a cylinder if } c > 0 \\ \text{a line if } c = 0 \\ \varnothing \text{ if } c < 0. \end{cases}$$

Ex $h : \mathbb{R}^3 \to \mathbb{R}^1$; $\quad h\begin{pmatrix} x \\ y \\ z \end{pmatrix} = z - (x^2 + y^2)$. $\quad$ If $c \in \mathbb{R}^1$, $h^{-1}(c)$ is the paraboloid $\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : z = x^2 + y^2 + c \right\}$.

Ex $F : \mathbb{R}^3 \to \mathbb{R}^2$; $\quad F\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x^2 + y^2 \\ z \end{pmatrix} \in \mathbb{R}^2$,

$$F^{-1}\begin{pmatrix} c \\ d \end{pmatrix} = \begin{cases} \text{a circle if } c > 0 \\ \text{a point if } c = 0 \\ \varnothing \text{ if } c < 0. \end{cases}$$

Ex $G : \mathbb{R}^3 \to \mathbb{R}^2$; $\quad G\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} z - (x^2 + y^2) \\ z + (x^2 + y^2) \end{pmatrix}$. Note that F and G have the same level sets. (The level sets are subsets of the domain.)

Note that the graph (a) is a special case of (b). An m-dimensional "surface" in $\mathbb{R}^N$ can be given 2 ways:

as the **image of** $\mathbb{R}^m$ under an appropriate map $\mathbb{R}^m \to \mathbb{R}^N$ OR
as the **preimage** of a point in $\mathbb{R}^{N-m}$ under an appropriate map $\mathbb{R}^N \to \mathbb{R}^{N-m}$.

Linear and Affine maps $\mathbb{R}^n \to \mathbb{R}^m$.

A **linear** map $L : \mathbb{R}^n \to \mathbb{R}^m$ is a map of the form

$$L\begin{pmatrix}x_1\\ \vdots\\ x_n\end{pmatrix} = \begin{pmatrix}a_{11}\dots a_{1n}\\ \vdots\\ a_{m1}\dots a_{mn}\end{pmatrix}\begin{pmatrix}x_1\\ \vdots\\ x_n\end{pmatrix}$$

$$a_{ij}\in\mathbb{R}$$

Example:

$$L:\mathbb{R}^3\to\mathbb{R}^2\ L\begin{pmatrix}x\\ y\\ z\end{pmatrix} = \begin{pmatrix}1&2&3\\4&5&6\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix} = \begin{pmatrix}x+2y+3z\\4x+5y+6z\end{pmatrix}.$$

An **affine map** $A:\underset{\vec{x}}{\mathbb{R}^m}\to\underset{\vec{y}}{\mathbb{R}^n}$ is a map of the form $A(\vec{x}) = L(\vec{x})+\vec{y}_0$, $\vec{y}_0\in\mathbb{R}^n$.

Example:

$$A:\mathbb{R}^3\to\mathbb{R}^2, A\begin{pmatrix}x\\ y\\ z\end{pmatrix} = \begin{pmatrix}1&2&3\\4&5&6\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix} + \begin{pmatrix}7\\8\end{pmatrix} = \begin{pmatrix}x+2y+3z+7\\4x+5y+6z+8\end{pmatrix}.$$

The **derivative** of a map $f:\mathbb{R}^n\to\mathbb{R}^m$ and **affine approximation**.

Unless specified, all maps will be assumed to be *smooth*, i.e. to have partial derivatives of all orders.

Let $f:\mathbb{R}^n{}_{\vec{x}}\to\mathbb{R}^m{}_{\vec{y}}$, $\vec{x}_0\in\mathbb{R}^n$. The **derivative** of f at $\vec{x}_0$ is

$$f'(\vec{x}_0) = \begin{pmatrix}\frac{\partial y_1}{\partial x_1}(\vec{x}_0)\dots\frac{\partial y_1}{\partial x_n}(\vec{x}_0)\\ \vdots\\ \frac{\partial y_m}{\partial x_1}(\vec{x}_0)\dots\frac{\partial y_m}{\partial x_n}(\vec{x}_0)\end{pmatrix}$$

This gives an affine approximation $A:\mathbb{R}^n\to\mathbb{R}^m$

$$A(\vec{x}) = f(\vec{x}_0) + f'(\vec{x}_0)(\vec{x}-\vec{x}_0).$$

"the best affine approximation to f near $x = x_0$. "

(if f has continuous partials, A is "best" in the sense that it is the **only** affine map with

$$\lim_{\vec{x}\to\vec{x}_0}\frac{A(\vec{x})-f(\vec{x})}{|\vec{x}-\vec{x}_0|} = \vec{0}.)$$

Exercise:

$$f:\mathbb{R}^2\to\mathbb{R}^3 \qquad f\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}x+y\\ x-y\\ 4xy\end{pmatrix}$$

a Show that the image of f lies on the surface $w = u^2 - v^2$.

b Find the affine approx to f near $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Identify the image of A.

c Estimate $f\begin{pmatrix} 1.01 \\ .98 \end{pmatrix}$ using A.

Solution:

a) Plug $u = x + y, v = x - y, w = 4xy$ into the equation $w = u^2 - v^2$. What happens? What does this tell you ?

b)

$$f\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 4 \end{pmatrix} \quad f' = \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ 4y & 4x \end{pmatrix}. \quad f'\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ 4 & 4 \end{pmatrix}$$

$$A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 4 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ 4 & 4 \end{pmatrix}\begin{pmatrix} x - 1 \\ y - 1 \end{pmatrix} = \begin{pmatrix} x + y \\ x - y \\ 4x + 4y - 4 \end{pmatrix}$$

The image of $A : \mathbb{R}^2 \to \mathbb{R}^3$ is a **plane** in $\mathbb{R}^3$, the plane containing the point $\begin{pmatrix} 2 \\ 0 \\ 4 \end{pmatrix}$ and the vectors $\begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix}$. What plane is this? Recall from Calc 3 how to find the **tangent plane** to the surface $w = u^2 - v^2$ at the point $\begin{pmatrix} 2 \\ 0 \\ 4 \end{pmatrix}$: The normal vector is

$$\nabla(w - u^2 + v^2) = \begin{pmatrix} -2u & 2v & 1 \end{pmatrix}$$

evaluated at the point $\begin{pmatrix} 2 \\ 0 \\ 4 \end{pmatrix}$, i.e.$\begin{pmatrix} -4 & 0 & 1 \end{pmatrix}$.

Note that $\begin{pmatrix} -4 & 0 & 1 \end{pmatrix}$ is perpendicular to the vectors $\begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix}$.

So the image of the affine map is the tangent plane to the image of f.

c)

$$f\begin{pmatrix} 1.01 \\ .98 \end{pmatrix} \approx \begin{pmatrix} 2 \\ 0 \\ 4 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ 4 & 4 \end{pmatrix}\begin{pmatrix} .01 \\ -.02 \end{pmatrix} = \begin{pmatrix} 1.99 \\ .03 \\ 3.96 \end{pmatrix}$$

$$(\textbf{Actual}: \begin{pmatrix} 1.99 \\ .03 \\ 3.9592 \end{pmatrix})$$

(Note that the first 2 entries in the affine approximation are exactly correct, but the third is not . Why is this?)

Chain Rule: $\mathbb{R}^n \xrightarrow{f} \mathbb{R}^m \xrightarrow{g} \mathbb{R}^P$

If f, g are continuous and differentiable, then so is $g \circ f$, and

$$(g \circ f)' = g' \bullet f'.$$

(The multiplicaton "$\bullet$" on the right is matrix multiplication! "$\circ$" is composition of functions.

Exercise:

$$f \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x^2 + y^2 \\ x^2 - y^2 \\ y^2 \end{pmatrix}; g \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} u + v + w \\ uv \end{pmatrix}$$

Use the chain rule to find $(g \circ f)' \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. Check your answer.

$$\begin{matrix} \mathbb{R}^2 & \xrightarrow{f} & \mathbb{R}^3 & \xrightarrow{g} \mathbb{R}^2 \\ \begin{pmatrix} 1 \\ 2 \end{pmatrix} & & \begin{pmatrix} 5 \\ -3 \\ 4 \end{pmatrix} & \end{matrix}$$

$$(g \circ f)' \begin{pmatrix} 1 \\ 2 \end{pmatrix} = g' \begin{pmatrix} 5 \\ -3 \\ 4 \end{pmatrix} \cdot f' \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ -3 & 5 & 0 \end{pmatrix} \begin{pmatrix} 2 & 4 \\ 2 & -4 \\ 0 & 4 \end{pmatrix} = \begin{pmatrix} 4 & 4 \\ 4 & -32 \end{pmatrix}$$

Check:

$$g \circ f \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x^2 + y^2 \\ x^4 - y^4 \end{pmatrix} \to (g \circ f)' \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 4x & 2y \\ 4x^3 & -4y^3 \end{pmatrix} |_{\begin{pmatrix} 1 \\ 2 \end{pmatrix}} = \begin{pmatrix} 4 & 4 \\ 4 & -32 \end{pmatrix}.$$

Exercise: Why matrix multiplication?? If a linear map L_1 is represented by the matrix M_1, and the linear map L_2 is represented by the matrix M_2, then the linear map $L_1 \circ L_2$ is represented by the matrix $M_1 \bullet M_2$. Spend a few minutes convincing yourself of this fact.

A little topology: Def. A *neighborhood* of a point $\vec{x}$ in $\mathbb{R}^n$ is a subset of $\mathbb{R}^n$ containing the open ball

$$B_{\vec{x}}(r) = \{\vec{y} \in \mathbb{R}^n : |\vec{y} - \vec{x}| < r\}$$

for some $r > 0$. An *open set* in $\mathbb{R}^n$ is a set that is a neighborhood of each of its points.

Theorem 1 (Inverse Function Theorem). *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be smooth , with $f(\vec{x}_0) = \vec{y}_0$. If $f'(\vec{x}_0)$ is invertible then f is bijective in a neighborhood U of $\vec{x}_0$. The restriction of f to U has a continuous smooth inverse $f^{-1} : V \to U$ with derivative*

$$(f^{-1})'(\vec{y}_0) = (f'(\vec{x}_0))^{-1}$$

(The inverse on the right is the inverse matrix.)
Example:

$$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x^2 + y^2 \\ x^2 - y^2 \end{pmatrix}. \quad f' = \begin{pmatrix} 2x & 2y \\ 2x & -2y \end{pmatrix}$$

$$\det f' = -8xy = 0 \Leftrightarrow x = 0 \text{ or } y = 0$$

Thus f is bijective and has a continuous inverse in a neighborhood of $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ **provided** $x_0 \neq 0$ **and** $y_0 \neq 0$.

a Find the affine map that best approximates f^{-1} close to $f\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ -3 \end{pmatrix}$.

b Estimate $\begin{pmatrix} x \\ y \end{pmatrix} \approx \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ if $x^2 + y^2 = 5.08$ and $x^2 - y^2 = -3.16$

Ans: $f'\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ 2 & -4 \end{pmatrix}$.

$$(f^{-1})'\begin{pmatrix} 5 \\ -3 \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ 2 & -4 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{1}{8} & -\frac{1}{8} \end{pmatrix} * .$$

Affine map

$$A\begin{pmatrix} u \\ v \end{pmatrix} = f^{-1}\begin{pmatrix} 5 \\ -3 \end{pmatrix} + (f^{-1})'\begin{pmatrix} 5 \\ -3 \end{pmatrix}\begin{pmatrix} u-5 \\ v+3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{1}{8} & -\frac{1}{8} \end{pmatrix}\begin{pmatrix} u-5 \\ v+3 \end{pmatrix}$$

In particular $f^{-1}\begin{pmatrix} 5.08 \\ -3.16 \end{pmatrix} \approx \begin{pmatrix} .98 \\ 2.03 \end{pmatrix}$.

Note in this case we could have solved for x, y in terms of u, v : $x = \sqrt{\frac{1}{2}(u+v)}, y = \sqrt{\frac{1}{2}(u-v)}$ but this is **not** always possible

* Inverse of a 2×2 matrix: $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{pmatrix}$.

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Def A *change of coordinates* is a smooth map f from an open set U in $\mathbb{R}^n$ to an open set V in $\mathbb{R}^n$ that is a *diffeomorphism,* that is a smooth bijection with a smooth inverse.

Example. A 2-dimensional coordinate change $\begin{pmatrix} x \\ y \end{pmatrix} \leftrightarrow \begin{pmatrix} u \\ v \end{pmatrix}$ is a local bijection with a differentiable inverse. We think of it like this: You can use these coordintes, or you can use those coordinates, but they are really just 2 different ways of labeling points in *"the same set". We take the point of view that changing coordinates is not really changing anything.*

Here's an example you know:

$$f \begin{pmatrix} r \\ \theta \end{pmatrix} = \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix} =: \begin{pmatrix} x \\ y \end{pmatrix}.$$

Then

$$f' = \begin{pmatrix} cos\theta & -r sin\theta \\ sin\theta & r cos\theta \end{pmatrix}$$

has $\det f' = r$.

So this is a good coordinate change in some neighborhood of any point $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ where $r \neq 0$, (i.e. except at the origin). (Why (because of which coordinate) is this an admissible coordinate change only locally, even if we exclude the origin?)

Example: A linear map $L : \mathbb{R}^n \to \mathbb{R}^n$ is an admissible coordinate change (on all of $\mathbb{R}^n$, *not* just locally) provided L is invertible.

Example: Consider the map $f : \mathbb{R}^3 \to \mathbb{R}^3$ by $f(\rho, \theta, \phi) = (\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho \cos\phi)$. This will be a good coordinate change at least locally provided the derivative f' is invertible, i.e. provided the determinant of f' is nonzero. Without computing any derivatives, write down the determinant of f'. (I think you know what it is!) Which points (ρ, θ, ϕ) have a neighborhood in which f is locally a good coordinate change?

Def: A (smooth) $k-$dimensional *manifold* is a subset M of $\mathbb{R}^N$ with the property that every point in M has a neighborhood U in $\mathbb{R}^N$ with local coordinates $(x_1, ..., x_N)$ so that

$$M \cap U = \{(x_1, ..., x_N) : x_{k+1} = ... = x_N = 0.$$

M is also called a *smooth* $k-$*dimensional surface in* $\mathbb{R}^N$. **Examples:........**Circle, helix, quadric surfaces except for....which one?

If $M \subset \mathbb{R}^N$ is a $k-$dimensional manifold and $P \in M$, then M has a *tangent space* $T_P M$ at the point P. It is a $k-$dimensional plane in $\mathbb{R}^N$ containing the point P. If M is a *curve* (a one-dimensional manifold *)* the tangent space is a *tangent line;* if M is a *surface* (a two-dimensional manifold, though higher dimensional manifolds are also sometimes called surfaces, or $k-$dimensional surfaces) the tangent space is a (2-)plane...

Theorem 2 (Immersion theorem). *Let* $f : \mathbb{R}^m \to \mathbb{R}^{n+m}$, *with* $f(\vec{x}_0) = \vec{y}_0$. *If* $f'(\vec{x}_0)$ *is injective, then* f *is* ***injective*** *in an open ball* U *about* $\vec{x}_0$*; the image of* $f|_U$ *is an* m*-dimensional manifold in* $\mathbb{R}^{n+m}$*; the* ***tangent space*** *to this surface at* $f(\vec{x}_0)$ *is the* ***image*** *of the affine map*

$$A(\vec{x}) = f(\vec{x}_0) + f'(\vec{x}_0)(\vec{x} - \vec{x}_0).$$

The map f *is called an* ***immersion*** *at* $\vec{x}_0$.

Theorem 3 (Submersion theorem). *Let* $f : \mathbb{R}^{n+k} \to \mathbb{R}^k$.

If $f'(\vec{x}_0)$ *is surjective at every point in the "level set"* $f^{-1}(\vec{y}_0)$*, then the level set is an* n*-dimensional manifold in* $\mathbb{R}^{n+k}$*. with* ***tangent space*** *at* $\vec{x}_0$ *given by*

$$f'(\vec{x}_0)(\vec{x} - \vec{x}_0) = \vec{0}.$$

If $f'(\vec{x}_0)$ *is surjective, the map* f *is called a* ***submersion*** *at* $\vec{x}_0$.

Remarks (1) *In each of the three theorems, the hypothesis (bijective, injective, surjective) is that the derivative has maximal rank.* ***(2)*** *The Maximal Rank theorem says that every* n*-dimensional manifold* M *in* $\mathbb{R}^{n+k}$ *can be "given" locally as the image of an immersion* $f : \mathbb{R}^n \to \mathbb{R}^{n+k}$*,and can also be "cut out" locally by* k *independent functions*

$$F_1(x_1, ..., x_{n+k}) = 0$$
$$...$$
$$F_k(x_1, ..., x_{n+k}) = 0.$$

Examples of Immersion and Submersion Theorems .

1. $f : \mathbb{R} \to \mathbb{R}^2, f(t) = \begin{pmatrix} t \\ t^2 \end{pmatrix}$. Show f is an immersion. What is the image of f ? Find a parametric form for the tangent line to the image of f at $f(2)$.

   ANS: $f' = \begin{pmatrix} 1 \\ 2t \end{pmatrix}$ which **always** has max rank $\Rightarrow$ immersion. Image is the parabola $y = x^2$, a 1-dimensional manifold (i.e. curve) in $\mathbb{R}^2$. Parametric tangent line:

   $$A(t) = f(2) + f'(2)(t-2) = \begin{pmatrix} 2 \\ 4 \end{pmatrix} + \begin{pmatrix} 1 \\ 4 \end{pmatrix}(t-2).$$

   The image of A is the line through $\begin{pmatrix} 2 \\ 4 \end{pmatrix}$ with direction vector $\begin{pmatrix} 1 \\ 4 \end{pmatrix}$.

2. Example: Write down a formula for an immersion whose image looks like the letter α. Check to see that it is an immersion. Does this contradict the immersion theorem? Why not?

Def Let $f : \mathbb{R}^n \to \mathbb{R}^m$. A point $\overrightarrow{x} \in \mathbb{R}^n$ is a *regular point* if $f'(\overrightarrow{x})$ exists and is surjective. Otherwise $\overrightarrow{x}$ is a *critical point* . A point $\overrightarrow{y} \in \mathbb{R}^m$ is a *critical value* if the level set $f^{-1}(\overrightarrow{y})$ contains a critical point; otherwise $\overrightarrow{y}$ is a *regular value* . The submersion theorem says that the level set of a *regular value* is a smooth manifold. Note that if $n < m$, all points and values are critical.

3. Example: $F \begin{pmatrix} x \\ y \end{pmatrix} = x^2 + y^2$

   a Sketch the level sets of F.

   b What are the regular points of F?

   c What are the regular values?

   d Find the tangent line to the level set S containing the point $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

   e Find a parametric form for the level set S.

f Find a parametric form for the tangent line, and check same as d.

ANS:

a

b $F' = (2x\ 2y)$ has max rank unless $x = y = 0$ Only critical point is $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$. All other points regular.

c Only critical value of F is $F\begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0$; other values regular.

Notes: $F^{-1}(c) = \begin{cases} \text{circle if } c > 0 \\ \text{pt if } c = 0 \\ \emptyset \text{ if } c < 0 \end{cases}$

Which is a nice smooth 1-dimensional surface for any c other than the critical value $c = 0$. (Submersion Theorem)

d $F'\begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} x-1 \\ y-0 \end{pmatrix} = 0$, i.e. $(1\ 0)\begin{pmatrix} x-1 \\ y-0 \end{pmatrix} = 0$ i.e. $x - 1 = 0$ Tangent line.

e $f(t) = \begin{pmatrix} cost \\ sint \end{pmatrix}$ is a parametric form for the level set

$$S = F^{-1}(F\begin{pmatrix} 1 \\ 0 \end{pmatrix}) = F^{-1}(1) = \{\begin{pmatrix} x \\ y \end{pmatrix} | x^2 + y^2 = 1\}$$

(note $cos^2 t + sin^2 t = 1 \forall t$)

Note f is an immersion since $f' = \begin{pmatrix} -sint \\ cost \end{pmatrix}$ which has rank 1 always (why?).

f Parametric form for tangent space:

$$A(t) = f(0) + f'(0)(t - 0) = \begin{pmatrix} 1 \\ t \end{pmatrix}$$

(the vertical line through the point $f(0)$.

Note f is a local injection but not an injection!

4. $F\begin{pmatrix} x \\ y \end{pmatrix} = xy$.

a Sketch the level sets of F.

b What are the regular points of F?

c What are the regular values?

d Looking at the level sets of F, what can you say about the critical points?

e Find a parameterization f for a piece of the level set $F^{-1}(1)$ that contains the point $\begin{pmatrix} -2 \\ \frac{-1}{2} \end{pmatrix}$.

f Find the tangent line to the level set $F^{-1}(1)$ at the point $\begin{pmatrix} -2 \\ \frac{-1}{2} \end{pmatrix}$ in parametric form using f, and in implicit form using F. Check with what you know from Calc 1!

ANS:

a $xy = c$

b $F' = (y \quad x)$ which has max rank (1) unless $x = 0$ and $y = 0$. So $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is only critical point.

c The only critical value is $F\begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0$.

Other values are regular values. Note: if c is a regular value as predicted by Submersion theorem (i.e. $c \neq 0$) $F^{-1}(c)$ is a 1-dimensional manifold.

d You can tell from the picture that $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is a critical point.

5. $f : \mathbb{R} \to \mathbb{R}^2 \quad f(t) = \begin{pmatrix} t^2 \\ t^3 \end{pmatrix}$. Is f an immersion?

Sketch the image of f, and discuss.

$F' = \begin{pmatrix} 2t \\ 3t^2 \end{pmatrix}$. This is an immersion at each point except at $t = 0$.

The image lies on the curve $x^3 = y^2$, which has no tangent line at $\begin{pmatrix} 0 \\ 0 \end{pmatrix} = f(0)$.

Note that f is a *smooth* function (all derivatives exist and are continuous!) but the **image** of f fails to be smooth since f is not an immersion. (What is the velocity when $t = 0$? What does this mean?)

Preimages can also be non-smooth for functions which **are** smooth, but whose derivative fails to be surjective. In fact (!) ANY closed subset of $\mathbb{R}^k$ is the zero set of some **smooth** (infinitely differentiable) $f : \mathbb{R}^k \to \mathbb{R}$. Since closed subsets of $\mathbb{R}^k$ can be pathological (think of $F \times F \subseteq \mathbb{R}^2$ where F is the Cantor set) Example 6 above is really very mild.

Example: The sphere $S^n = \{ \begin{pmatrix} x_0 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^{n+1} | x_0^2 + x_1^2 + \ldots x_n^2 = 1 \}$.

Let $F \begin{pmatrix} x_0 \\ \vdots \\ x_n \end{pmatrix} = x_0^2 + \ldots + x_n^2$. Then $F' = (2x_0 \ldots 2x_n)$ which has max rank unless $x_0 = \ldots = x_n = 0$. So 1 is a regular value, and S^n is a smooth n-dimensional surface in $\mathbb{R}^{n+1}$. S^1 is a circle in $\mathbb{R}^2$, and S^2 is an "ordinary sphere" in $\mathbb{R}^3$.

Exercise: Discuss the following parameterizations of the sphere $x^2 + y^2 + z^2 = 1$. On which part of the sphere is each a valid coordinate system? Where do these parameterizations come from? hint: Each comes from using 2 of the standard coordinates (cartesian, cylindrical or spherical) as parameters.

a $\begin{pmatrix} u\ cosv \\ u\ sinv \\ \sqrt{1-u^2} \end{pmatrix}$

b $\begin{pmatrix} u \\ v \\ \sqrt{1-u^2-v^2} \end{pmatrix}$

c $\begin{pmatrix} cos(u)\ sin(v) \\ sin(u)\ sin(v) \\ cos(v) \end{pmatrix}$

d $\begin{pmatrix} u \\ -\sqrt{1-u^2-v^2} \\ v \end{pmatrix}$

Exercise:

Find local coordinates on S^2 near the north pole $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = N$. Look at the restriction of the function $G \begin{pmatrix} x \\ y \\ z \end{pmatrix} = z$ to the sphere near N. What kind of function is it? Does it have critical points? What do its level sets look like?

Solution:

To find local coordinates means we must find a parametric form for the surface near N. The only one of the above parameterizations that is good near N is (b). Specifically let

$$g : \{ \begin{pmatrix} u \\ v \end{pmatrix} | u^2 + v^2 < 1 \} \to \text{(open) Upper hemisphere in} S^2$$

$$g\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u \\ v \\ \sqrt{1-u^2-v^2} \end{pmatrix}.$$

Then

$$g' = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{-u}{\sqrt{1-u^2-v^2}} & \frac{-v}{\sqrt{1-u^2-v^2}} \end{pmatrix}.$$

has maximal rank at each $\begin{pmatrix} u \\ v \end{pmatrix}$ in the open disk $u^2 + v^2 < 1$. So u, v are local coordinates on the (open) upper hemishphere.

In terms of local coordinates $G(u, v) = \sqrt{1-u^2-v^2}$.

$G' = \left(\frac{-u}{\sqrt{1-u^2-v^2}}, \frac{-v}{\sqrt{1-u^2-v^2}}\right)$, so G has a critical point at $\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \leftrightarrow N$.

The level sets are the sets $\sqrt{1-u^2-v^2} = const$ which are the same as the level sets of $u^2 + v^2$:

The point at the center of the u, v coordinate system is N, the critical point.

Note: On $\mathbb{R}^3, G$ has **no** critical points. The level sets of G on $\mathbb{R}^3$ are the planes $z = const$. The level sets of G on the sphere are the intersections of these planes with the sphere. The function looks very different on the sphere.

Exercise: Sketch the level sets and locate the critical points. Find a parameterization of the level set of a regular value. (That is, an immersion onto the level set.) Describe the level set of a critical value. (Can you tell by looking at this level set that the value is critical?)

a) $G : \mathbb{R}^3 \to \mathbb{R}^1$ by $G\begin{pmatrix} x \\ y \\ z \end{pmatrix} = x^2 + y^2$.

b) $H : \mathbb{R}^3 \to \mathbb{R}^1$ by $H\begin{pmatrix} x \\ y \\ z \end{pmatrix} = x - cos(z)$.

Now let $K : \mathbb{R}^3 \to \mathbb{R}^2$ by $K\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x^2 + y^2 \\ x - cos(z) \end{pmatrix}$.

c) Find the critical points of K. For which $R > 0$ is $\begin{pmatrix} R^2 \\ 0 \end{pmatrix}$ a critical value?

Describe the level set $K^{-1}\begin{pmatrix} R^2 \\ 0 \end{pmatrix}$.

d) if $R^2 > 1$

e) if $0 < R^2 < 1$

f) if $R = 1$.

Partial Solution

c) First we find the critical points of K. We have

$$K' = \begin{pmatrix} 2x & 2y & 0 \\ 1 & 0 & \sin z \end{pmatrix}.$$

The point $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ will be a critical point if and only if Row 1=λ(Row 2) for some $\lambda \epsilon \mathbb{R}$, or Row 2=$\tau$(Row 1) for some $\tau \epsilon \mathbb{R}$. In the second case $\tau \neq 0$ (because of the 1 in Row 2) so putting $\lambda = 1/\tau$ we see that $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ will be a critical point if and only if Row 1=λ(Row 2) for some $\lambda \epsilon \mathbb{R}$. This is true $\Leftrightarrow$

$$2x = \lambda \text{ AND } 2y = \lambda \cdot 0 \text{ AND } 0 = \lambda \cdot \sin z$$

i.e. $\Leftrightarrow$

$$y = 0 \text{ AND } 0 = x \cdot \sin z$$

i.e. $\Leftrightarrow$

$$y = 0 \text{ AND (} x = 0 \text{ OR } z = m\pi)$$

The critical *points* are thus of 2 types:

$$x = y = 0 \quad \text{(points on the z-axis)}$$

and

$$y = 0, \ z = m\pi \ \text{ (points on a horizontal line } z = m\pi \text{ in the } xz\text{plane)}$$

If $R > 0$, and $\begin{pmatrix} R^2 \\ 0 \end{pmatrix}$ is a critical *value*, then there is a critical *point* $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ with $x^2+y^2 = R^2$, AND $x - cos(z) = 0$. It is easy to see that this critical point must be of the second type (since for critical points of the first type, $x^2+y^2 = 0$), and also that $y = 0$, $z = m\pi$, and $x = cos(z) = \pm 1$. Thus if $R > 0$, $\begin{pmatrix} R^2 \\ 0 \end{pmatrix}$ is a critical value if and only if $R = 1$.

Next we will look at the level sets $K^{-1}\begin{pmatrix} R^2 \\ 0 \end{pmatrix}$ and see how the computation we just did compares with the geometry. We are looking at the intersection of a cylinder with a "corrugated plane." What this looks like depends on R, i.e. on the radius of the cylinder relative to the size of the wiggles in the plane.

d) If R is very large, and if we step back, the intersection will look like the intersection of a cylinder with a slightly wiggly plane along the axis of the cylinder. (*Figure 1)* In this case it looks as if we should be able to parameterize each piece of the intersection with the the coordinate z; thus we will try to solve for x and y in terms of z using the equations

$$x^2 + y^2 = R^2$$
$$x = cos(z)$$

The reader should check the following: If $R > 1$, the level set $K^{-1}\begin{pmatrix} R^2 \\ 0 \end{pmatrix}$ has 2 connected components, one where $y > 0$ and the other where $y < 0$. In either component we can use z as a parameter, and

$$f_{\pm} : \mathbb{R} \to \mathbb{R}^3$$
$$f_{\pm}(t) = < cost, \pm\sqrt{R^2 - cos^2 t}, t >$$

is an immersion onto a component. Where did this formula come from?? (Hint: "$t = z$")The components look like 2 wavy stripes traveling the length of the cylinder $G^{-1}(R^2)$. (*Figure 1)* Note by our computation above, if $R > 1$, the level set $K^{-1}\begin{pmatrix} R^2 \\ 0 \end{pmatrix}$ is the level set of a regular value and thus by the Submersion Theorem it is a smooth surface of dimension 3-2=1. **Interesting question:** Can we use the same parameterization if $R = 1$? What is the image of f_+ if $R = 1$?? Is it an immersion?

e) If R^2 is very small and we zoom in, the part of the corrugated plane that meets the cylinder will look like a family of planes intersecting the cylinder axis at an angle $\pi/4$:.*(Figure 2)*

<<Remark and explanation of the angle $\pi/4$: On this scale the corrugated plane is very well approximated by its tangent plane at the point of intersection. At the point of intersection x and y are very close to 0 (because $x^2 + y^2 = R^2$ which is very small) and because $x = cos(z)$, z is close to $\frac{2m+1}{2}\pi$ for some $m\epsilon\mathbb{Z}$. Thus the tangent plane to the corrugated plane at such a point $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ (with $a \approx 0$, $b \approx 0$, $c \approx \frac{2m+1}{2}\pi$) is given implicitly by

$$\begin{pmatrix} 1 & 0 & \sin z \end{pmatrix} \begin{pmatrix} x - a \\ y - b \\ z - c \end{pmatrix} = 0$$

If R^2 is very small this is approximately the plane

$$\begin{pmatrix} 1 & 0 & \pm 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z - \frac{2m+1}{2}\pi \end{pmatrix} = 0$$

with the sign $+1$ if m is even and -1 if m is odd. This plane is the translation parallel to the $y-$ axis of the line $x = \mp(z - \frac{2m+1}{2}\pi)$. >>

In the case $0 < R^2 < 1$ it looks from the picture as if we should be able to use the cylindrical coordinate Θ as a parameter on each component; thus we will try to solve for x, y, and z in terms of Θ.

The reader should check the following: If $0 < R < 1$, for each $m \in \mathbb{Z}$, and each choice of sign,

$$g_m^{\pm} : \mathbb{R} \to \mathbb{R}^3$$
$$g_m^{\pm}(t) = < R\cos t, R\sin t, \pm cos^{-1}(R\cos t) + 2m\pi >$$

is an immersion onto a component. (Why is $cos^{-1}(R\cos t)$ a differentiable function of t? Use the inverse function theorem carefully. You will need the fact that $0 < R < 1$.) Where did the formula for $g_m^{\pm}$ come from ?? (Hint: "$t = \Theta$".)

There is one component for each $m \in \mathbb{Z}$ and each choice of sign. Note $0 < \cos^{-1}(R\cos t) < \pi$ so the different components do not intersect. Recall from our earlier computation that if $0 < R < 1$, the level set $K^{-1}\begin{pmatrix} R^2 \\ 0 \end{pmatrix}$ is the level set of a regular value and thus by the Submersion Theorem it is a manifold of dimension 3-2=1. **Interesting question**: Can we use the same parameterization if $R = 1$? What is the image of g_m^{+} if $R = 1$?? Is it an immersion?

f) If $R = 1$, the level set consists of two helices (each parameterized by z):

$$t \mapsto < cost, sint, t >$$

and

$$t \mapsto < cost, -sint, t >$$

intersecting at the points $< 1, 0, 2m\pi >$ and $< -1, 0, (2m+1)\pi >$. Note that the points of intersection are precisely the critical points of K on this level set computed above: $y = 0$, $z = m\pi$, and $x = \pm 1$. Note that $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is not a regular value, so we expect that $K^{-1}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ might not be a smooth surface of dimension $3 - 2 = 1$.

Exercise: What do the level sets $K^{-1}\begin{pmatrix} R^2 \\ c \end{pmatrix}$ look like if $c \neq 0$? How do all the level sets fit together to fill up $\mathbb{R}^3$?

Example If $M \subset \mathbb{R}^N$ is a manifold, the tangent space $TM = \{(P, \mathbf{V}) : P \in M$ and $\mathbf{V} \in T_P M\}$ is a manifold. What is the dimension? If M is given locally as the image of an immersion f, express TM locally as the image of an immersion. If M is the level set of a submersion F, express TM as the level set of a submersion.

Example: The Special Linear Group

Let $M(n)$ be the set of $n \times n$ matrices. Note that $M(n)$ is identified with the Euclidean space R^{n^2}, since an $n \times n$ matrix has n^2 entries. The special linear group $Sl(n) =: Sl(n, \mathbb{R})$ is the set of $n \times n$ matrices with determinant 1. To view $Sl(n)$ as a surface, Let $F : R^{n^2} \to \mathbb{R}$ by $F(A) = \det A$. Then $Sl(n) = F^{-1}(1)$.

Is 1 a regular value? The determinant is a homogeneous polynomial of degree n in the variables $x_1, ...x_{n^2}$, that is $\det(tA) = t^n \det A$. We use Euler's identity for a homogeneous polynomial p of degree n:

$$\sum x_i \frac{\partial p}{\partial x_i} = np \qquad (*)$$

Now

$$F' = (\frac{\partial F}{\partial x_1} \ \frac{\partial F}{\partial x_2} .... \frac{\partial F}{\partial x_{n^2}})$$

which has maximal rank at a point $(x_1, ..., x_{n^2})$ unless all the entries $\frac{\partial F}{\partial x_i}$ are 0. In this case $\sum x_i \frac{\partial F}{\partial x_i} = 0$; by (*) it follows that $np(x_1, ..., x_{n^2}) = 0$ and thus that $p(x_1, ..., x_{n^2}) = 0$. So 0 is the only critical value, and $Sl(n) = F^{-1}(1)$ is a smooth surface of dimension $n^2 - 1$ in R^{n^2}. It is noncompact for $n > 1$. (Why? Note that $Sl(n)$ contains the matrix $\begin{pmatrix} r & 0 & 0 & 0 & ... \\ 0 & \frac{1}{r} & 0 & 0 & \\ 0 & 0 & 1 & 0 & \\ 0 & 0 & 0 & 1 & \\ ... & & & & ... \end{pmatrix}$). In fact $Sl(1)$ is the point $1 \epsilon \mathbb{R}^1$; $Sl(2)$ can be identified with a solid torus (a 3-dimensional surface) in $\mathbb{R}^4$. Becasue it is a manifold, we can do calculus on it, using local coordinates as in the following exercise.

$Sl(n)$ is a *Lie Group*, i.e. a set that is both a group (under matrix multiplication) and a manifold for which the group operations (muliplication and inverse) are differentiable in local coordinates. Thus we can do algebra and geometry and topology at once.

Exercise: The purpose of this exercise is to understand the level sets of the trace function Tr on the Lie group $Sl(2)$.

Warm-up and motivation: Because the quadratic polynomial $\det(X - tI)$ always has 2 complex roots, every 2×2 matrix X has 2 complex eigenvalues (which might or might not be real). Show that in $Sl(2)$ the trace determines the eigenvalues. Moreover if $|Tr(X)| > 2$, the eigenvalues of X are real (either both positive or both negative); if $|Tr(X)| < 2$, then the eigenvalues are of the form $e^{i\theta}$ and $e^{-i\theta}$. (Hint: The determinant is the product of the eigenvalues, and the trace is the sum of the eigenvalues.)

<<Advanced warm-up and motivation: Show that the trace function is conjugation invariant ; that is, $Tr(A^{-1}XA) = Tr(X)$ for any n by n matrices A, X with A invertible. (Hint: Show that $Tr(X)$ is one of the coefficients of the characteristic polynomial of X. Then show that the characteristic polynomial

is conjugation invariant.) Thus the conjugacy class of X lies on the level set (for the function Tr) of X.>>

Let $F:\mathbb{R}^4\to\mathbb{R}, F\begin{pmatrix}a\\b\\c\\d\end{pmatrix}=ad-bc.$

a) Show that 1 is a regular value of F; thus $F^{-1}(1)$ is a 3-dimensional manifold S in $\mathbb{R}^4$.

Note: $F^{-1}(1)$ can be identified with the group $S\ell(2)$.

b) Show that $g:\mathbb{R}^3\to\mathbb{R}^4$ defined by

$$g\begin{pmatrix}u\\b\\c\end{pmatrix}=\begin{pmatrix}u+\sqrt{1+bc+u^2}\\b\\c\\-u+\sqrt{1+bc+u^2}\end{pmatrix}$$

is an immersion near $\begin{pmatrix}0\\0\\0\end{pmatrix}\in\mathbb{R}^3$ with image in S.

Thus $u=\frac{a-d}{2}, b$ and c are local coordinates on S near the identity matrix $I=:\begin{pmatrix}1\\0\\0\\1\end{pmatrix}\in S.$

c) (Digression and Review problem) Find the tangent space to S at I in parametric form using g, and in implicit form using F. Check they are the same.

d) Make a change of coordinates $b=v-w, c=v+w$ (why is this allowed?) to get coordinates $\begin{pmatrix}u\\v\\w\end{pmatrix}$ on S near I. Sketch the level sets of the **trace** function $T:S\to\mathbb{R}, T\begin{pmatrix}a\\b\\c\\d\end{pmatrix}=a+d$ in the u,v,w coordinate system near I.

[Hint: Show that in terms of the local coordinates $T=2\sqrt{1+v^2+u^2-w^2}$. Note T has the same level sets as $v^2+u^2-w^2$.

e) Locate I in the picture. What are the regular values of T near I? (Use local coordinates u,v,w).

Solution:

a) $f:\mathbb{R}^4\to\mathbb{R}$ by $f\begin{pmatrix}a\\b\\c\\d\end{pmatrix}=ad-bc.$ Then $f'=\begin{pmatrix}d & -c & -b & a\end{pmatrix}$ has maximal rank unless $a=b=c=d=0$, in which case $f=0$. Thus 0 is the only critical value, and $S=f^{-1}(1)$ is a smooth 3-dimensional surface in $\mathbb{R}^4$.

Remark: **This example shows the magic of the SubmersionTheorem. We can't "see" the surface $S = f^{-1}(1)$, but we can tell it is smooth surface because 1 is a regular value!!.**

<<Remark. We could get local coordinates on S near the point $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ by solving for (say) d in terms of $a, b, c.$: $d = \frac{1+bc}{a}$. (Note that $a \neq 0$ near the point $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ so this makes sense.) This would give an immersion (near the point $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$)

$$\begin{pmatrix} s \\ t \\ r \end{pmatrix} \mapsto \begin{pmatrix} s \\ t \\ r \\ \frac{1+tr}{s} \end{pmatrix}$$

In terms of the local coordinates s, t, r the trace function is

$$Trace = a + d = s + \frac{1+tr}{s}$$

and thus the level sets of the trace function are the sets where

$$s + \frac{1+tr}{s} = \text{constant}$$

These level sets are quadric surfaces in $\mathbb{R}^3$. (Why?) But they are not in standard form. Which quadric surfaces are they?? We will introduce the coordinates u, b, c and eventually u, v, w to render the level sets more recognizable. >>

b) For any point in the image of g, $ad - bc = (u + \sqrt{\ })(-u + \sqrt{\ }) - bc = 1$. So image $g \subseteq S$. And g is an immersion near $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ since

$$g' \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 + \frac{u}{\sqrt{\ }} & \frac{c}{2\sqrt{\ }} & \frac{b}{2\sqrt{\ }} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 + \frac{u}{\sqrt{\ }} & \frac{c}{2\sqrt{\ }} & \frac{b}{2\sqrt{\ }} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}$$

which has maximal rank since the columns are linearly independent. **Thus** (u, b, c) **are good local coordinates on the surface S near the point I.**

c) Tangent space to S at I is the image of the affine map

$$A\begin{pmatrix}u\\b\\c\end{pmatrix}=\begin{pmatrix}1\\0\\0\\1\end{pmatrix}+\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\\-1&0&0\end{pmatrix}\begin{pmatrix}u\\b\\c\end{pmatrix}=\begin{pmatrix}1+u\\b\\c\\1-u\end{pmatrix}.$$

to get the implicit form of tangent space: $F'(I)=(1\ 0\ 0\ 1)$ so equation is $(1\ 0\ 0\ 1)\begin{pmatrix}a-1\\b\\c\\d-1\end{pmatrix}=0$, i.e. $a-1+d-1=0$, or $a+d=2$ (a 3-dimensional plane in $\mathbb{R}^4$)

It's easy to see that $\begin{pmatrix}1+u\\b\\c\\1-u\end{pmatrix}$ lies on this plane for all u,b,c ; thus it's the same as the implicit form.

d) The further coordinate change

$$(u,b,c)\leftrightarrow(u,v,w)$$

is allowed (near the point I) because it is given by the smooth (linear) map $Q:\mathbb{R}^3\longrightarrow\mathbb{R}^3$

$$Q\begin{pmatrix}u\\v\\w\end{pmatrix}=\begin{pmatrix}1&0&0\\0&1&-1\\0&1&1\end{pmatrix}\begin{pmatrix}u\\v\\w\end{pmatrix}=\begin{pmatrix}u\\b\\c\end{pmatrix}$$

whose derivative has maximal rank at every point . **Thus (u,v,w) are good local coordinates on the manifold S near the point I.** Next we will express the trace function in terms of these local coordinates. Note $bc=v^2-w^2$; thus the trace

$$T=a+d=2\sqrt{1+bc+u^2}=2\sqrt{1+u^2+v^2-w^2}$$

is constant when

$$u^3+v^2-w^2=const.$$

The level sets of the trace function (near I) are the quadric surfaces $u^3+v^2-w^2=const.$

e) The cone is the level set Trace = 2, with the identity I at the vertex (because the vertex . $\begin{pmatrix}u\\v\\w\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$ corresponds to the point $\begin{pmatrix}a\\b\\c\\d\end{pmatrix}=\begin{pmatrix}1\\0\\0\\1\end{pmatrix}$).

A point on this cone corresponds to a matrix $X=\begin{pmatrix}a&b\\c&d\end{pmatrix}$ with $\det X=1$ and $traceX=2$. The cone "is" the space of matrices $X\epsilon S\ell(2)$ where 1 is an eigenvalue. (See warmup-motivatoin above.)

$$T' = (\frac{u}{\surd} \ \frac{v}{\surd} \ \frac{-w}{\surd})$$

which has maximal rank except at the origin. So at least near I, the only critical point of the trace function on $S\ell(2)$ is $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$. The only critical value "in our picture" is 2. , corresponding to the level set which is a cone. This is also the only level set in our picture that is not a smooth surface of dimension 3-1=2.

<<Remark: The matrix $-I$ is also a critical point of the restriction of *trace* to $S\ell(2)$, but our local coordinates do not extend over to that part of $S\ell(2)$. How much of the group $S\ell(2)$ is "covered" (parameterized) by our local coordinates u, v, w?>>

Remark The trace function T on $\mathbb{R}^4 = \left\{ \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \right\}$ has **no** critical points. The level sets of the trace function on $\mathbb{R}^4$ are boring! (3D planes) But the restriction of the trace function to the surface $S\ell(2)$ has critical points and interesting level sets.

Orthogonal group

Another important example is the orthogonal group

$$O(n) = \{A \in M(n) | AA^t = I\}$$

(You can check that this is a group. I is the $n \times n$ identity matrix.) $O(n)$ is also a smooth surface as follows:

$$\text{Let } F : \underset{\|_{\mathbb{R}^{n^2}}}{M(n)} \to S(n) = \textbf{Symmetric } n \times n \text{ matrices} = \mathbb{R}^{n(n+1)/2}$$

be defined by $F(A) = AA^t$.

Then $O(n) = F^{-1}(I)$, so we need to show that I is a regular value. Now $F'(A)$ is a matrix which represents a linear map $\mathbb{R}^{n^2} \to \mathbb{R}^{n(n+1)/2}$. It has max rank $\Leftrightarrow$ this linear map is surjective. Now

$$F'(A)(X) = \underset{h \to 0}{lim} \frac{F(A + hX) - F(A)}{h}$$

$$= \underset{h \to 0}{lim} \frac{(A + hX)(A^t + hX^t) - AA^t}{h}$$

$$= XA^t + AX^t.$$

Note: X, A both $n \times n$ matrices, thought of as points in $\mathbb{R}^{n^2}$. $f'(A)$ is an $n^2 \times n^2$ matrix.

Given $C \in \mathbb{R}^{n(n+1)/2}$, a symmetric matrix, let $X = \frac{1}{2}CA$. Then $XA^t + AX^t = \frac{1}{2}(C + C^t) = C$. Thus $F'(A)$ is surjective $\forall A \in O(n) \Rightarrow O(n) = F^{-1}(I)$ is a smooth surface in $\mathbb{R}^{n^2}$ of dimension $\frac{n(n-1)}{2}$.

Examples

$O(1) = \{-1, 1\}$ (multiplicative group) (2 points)

$O(2)$ is a non commutative group and a surface of dimension 1 in $\mathbb{R}^4$. As a surface it looks like 2 circles.

Every matrix in $O(2)$ can be written uniquely as

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^r \cdot \begin{pmatrix} cos\Theta & sin\Theta \\ -sin\Theta & cos\Theta \end{pmatrix}$$

Where $r = 0$ or 1 and $0 \leq \Theta < z\pi$.

$det = 1$ when $r = 0$ and $det = -1$ when $r = 1$.

Can you see how to parameterize the two circles?

$O(3)$ is a non commutative group and a surface of dimension 3 in $\mathbb{R}^9$. As a surface it looks like 2 copies of a compact 3-dimensional surface $SO(3)$ (which is **not** the 3 sphere S^3)..

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If we identify the $n \times n$ matrices $M(n)$ with the linear maps $\mathbb{R}^n \to \mathbb{R}^n$ in the usual way, then $O(n)$ is (identified with) the set of linear maps $A : \mathbb{R}^n \to \mathbb{R}^n$ that preserve the inner product on $\mathbb{R}^n$, i.e. $A \epsilon O(n)$ if and only if

$$\langle A\mathbf{x}, A\mathbf{y} \rangle = \langle \mathbf{x}, \mathbf{y} \rangle \quad (*)$$

for all $\mathbf{x}, \mathbf{y} \epsilon \mathbb{R}^n$. It follows that $O(n)$ is the set of rigid motions of $\mathbb{R}^n$ that fix the origin.

The group $O(n)$ can be identified with the set of orthomormal bases in $\mathbb{R}^n$. Let $\widehat{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\widehat{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ etc. If $A \epsilon O(n)$, then $A_{\widehat{i}}$ =First column of A; $A_{\widehat{j}}$ =Second column of A etc. Because A is a rigid motion (i.e.(*)), the columns of A are an orthonormal basis for $\mathbb{R}^n$. As a consequence $A \epsilon O(n)$ if and only if the columns of A are an orthonormal basis for $\mathbb{R}^n$.

Exercise: Show that the group $O(n)$ always has at least 2 connected components. Hint: Use the fact that $\det A = \det A^t$ and the fact that det is a group homomorphism to show that every orthogonal matrix has determinant ± 1. (In fact $O(n)$ has exactly two components for each n. If $\det A > 0$, then the linear map associated with the matrix A preserves orientation.)

Exercise: Show that the group $O(n)$ is compact for every n.

Example: The 3 dimensional surface $SO(3)$ (the component of $O(3)$ where $\det = 1$) can be identified with another surface, the *unit tangent space* T_1S^2 to the sphere S^2. This is the space of vectors $\mathbf{v}$ where $\mathbf{v}$ is a unit vector tangent to the sphere at some point $\mathbf{u}\epsilon S^2$. In other words

$$T_1S^2 \approx \{(\mathbf{u},\mathbf{v})\epsilon\mathbb{R}^3\times\mathbb{R}^3 : |\mathbf{u}| = |\mathbf{v}| = \mathbf{1} \text{ and } \mathbf{u}\cdot\mathbf{v} = 0\}.$$

(Show this is a smooth surface! What dimension?) To get the identification of T_1S^2 with $SO(3)$, we identify $(\mathbf{u},\mathbf{v})$ with the matrix A whose columns are $\mathbf{u},\mathbf{v}$, and $\mathbf{u}\times\mathbf{v}$.

More generally, if S is a smooth compact 2 dimensional surface in $\mathbb{R}^3$, we can get a smooth compact 3 dimensional surface T_1S in $\mathbb{R}^3\times\mathbb{R}^3$, the unit tangent space to S: the space of pairs $(\mathbf{r}, v)\in\mathbb{R}^3\times\mathbb{R}^3$ where $\mathbf{r}$ is (the position vector of) a point in S, and v is a unit vector tangent to S at $\mathbf{r}$.

Appendix: Linear Algebra

Facts about **determinants**

$Det : \mathbb{R}^{n^2} \to \mathbb{R}$

1. Det is a group homomorphism from the group of invertible $n\times n$ matrices (with matrix multiplication) to the group of non zero real numbers (with multiplication). In other words,

$$Det(A \bullet B) = Det(A)\cdot Det(B)$$

(Note: the multiplication on the left is matrix multiplication. The multiplication on the right is multiplication of real numbers.) And

$$Det(I) = 1$$

where

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \text{identity matrix}$$

2. Det is conjugation invariant, i.e. $Det(A^{-1}BA) = DetB$ (this follows from 1 above; do you see why?)

3. $Det(A) = 0 \Leftrightarrow$ the rows of A are linearly dependent
$\Leftrightarrow$ the columns of A are linearly dependent
$\Leftrightarrow$ the associated linear map $\mathbb{A}:\mathbb{R}^n\to\mathbb{R}^n$ is **not** injective
$\Leftrightarrow$ the associated linear map $\mathbb{A}:\mathbb{R}^n\to\mathbb{R}^n$ is **not** surjective
$\Leftrightarrow$ 0 is an eigenvalue of A

EIGENVALUES

λ is an eigenvalue of $A \Leftrightarrow A\overline{v} = \lambda\overline{v}$ for some $\overline{v}\in\mathbb{R}^n, \overline{v}\neq 0$
$\Leftrightarrow (A-\lambda I)\overline{v} = 0$
$\Leftrightarrow det(A-\lambda I) = 0$

(What does it mean geometrically if $A\overline{v} = \lambda\overline{v}$? What is the significance of the sign?)

Let A be an $n \times n$ matrix.

The characteristic polynomial $P(t) = det(A - tI)$ is a polynomial of degree n. Ex: If $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, then

$$P(t) = det \begin{pmatrix} 1-t & 2 \\ 3 & 4-t \end{pmatrix} = t^2 - 5t - 2.$$

FACT: A always satisfies its characteristic polynomial, i.e. $P(A) = 0$. In above example you can check

$$A^2 - 5A - 2 = 0$$

If $P(t)$ has n distinct (real) roots, then $\mathbb{R}^n$ has a basis of (real) **eigenvectors** for A. That is, we can find a basis $\overline{v}_1, \overline{v}_2, \ldots \overline{v}_n$ for $\mathbb{R}^n$ with

$$A\overline{v}_i = \lambda_i \overline{v}_i$$

for each i, where the eigenvalues λ_i are the roots of the characteristic polynomial. In this case A is **conjugate** to a diagonal matrix, i.e. there is a matrix B (invertible) with $B^{-1}AB = D$ with D diagonal; in fact $D = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & \ldots \\ & & & \ddots \lambda_n \end{pmatrix}$.

The columns of B are the eigenvectors $\overline{v}_1, \ldots, \overline{v}_n$.

Special matrices.

$O(n)$ = Orthogonal Group

$= \{A | A^t A = I\}$

A is **orthogonal** $\Leftrightarrow A^t A = I \Leftrightarrow$ the rows of A form an orthonormal basis for $\mathbb{R}^n$

$\Leftrightarrow$ the columns of A form an orthogobasis for $\mathbb{R}^n$.

$\Leftrightarrow$ The linear map $A : \mathbb{R}^n \to \mathbb{R}^n$ preserves the inner product $< \, , \, >$ on $\mathbb{R}^n$, i.e. $< \overline{x}, \overline{y} > = < A\overline{x}, A\overline{y} > \forall \overline{x}, \overline{y}$

$\Leftrightarrow$ The image of the standard orthonormal basis $\begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \end{pmatrix}, \ldots, \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$ of $\mathbb{R}^n$ is an orthonormal basis for $\mathbb{R}^n$.

Exercise: Show that $O(n)$ is a group, with the operation of matrix multiplication. .

$SO(n)$=Special Orthogonal Group

$= \{A | A^t A = I$ and $\text{Det} A = 1\}$

A is **symmetric** if $A^t = A$.

FACT: If A is symmetric, then all eigenvalues are real and A is **conjugate** by an **orthogonal** matrix Θ to a **diagonal** matrix $D = \begin{pmatrix} \lambda_1 & 0 & 0 & \\ 0 & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}$.

That is, $\Theta^{-1} A \Theta = D$.

This means that there is an **orthonormal basis** $\overline{v}_1, \ldots, \overline{v}_n$ for $\mathbb{R}^n$ consisting of eigenvectors for A. The vectors $\overline{v}_1$ are the columns of Θ.

This means that the linear map $\mathbb{A}$ does just what you would expect a diagonal map $\mathbb{D}$ to do.

Remarks 1. The roots of the characteristsic polynomial are not always real. For example the rotation matrix $\begin{pmatrix} cos\Theta & -sin\Theta \\ sin\Theta & cos\Theta \end{pmatrix}$ has eigenvalues $cos(\Theta) \pm isin\Theta$. Note this matrix is not symmetric!

2. A matrix **can** have all roots of the characteristic polynomial real, but still not be "diagonalizeable" (conjugate to a diagonal matrix). Example: The matrix $A = \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix}$ has characteristic polynomial $(t-1)^2$ which has 1 as a double root; however there is only one eigenvector, namely $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. (Geometrically A is a "shear" map). However: if the characteristic polynomial has n **distinct** real roots, then A is diagonalizeable. Moreover, for all A in an open dense set of $\mathbb{R}^{n^2}$, the roots of $P(t)$ are distinct (though not necessarily real) and thus A is conjugate to a diagonal matrix (may need complex numbers). Example: $\begin{pmatrix} cos\Theta & -sin\Theta \\ sin\Theta & cos\Theta \end{pmatrix}$ is a conjugate to $\begin{pmatrix} e^{iO} & 0 \\ 0 & e^{-i\Theta} \end{pmatrix}$

Exercises: Linear Algebra and Topology

1a Find the eigenvalues of the matrix $A = \begin{pmatrix} 0 & 10 \\ 10 & -15 \end{pmatrix}$ (solve $det(A - tI) = 0$)

b Check that $detA = \lambda_1 \lambda_2$ and $trA = \lambda_1 + \lambda_2$.

c Find an eigenvector $\begin{pmatrix} x \\ y \end{pmatrix}$ for each eigenvalue by solving $A \begin{pmatrix} x \\ y \end{pmatrix} = \lambda_i \begin{pmatrix} x \\ y \end{pmatrix}$ for each of the eigenvalues λ_i From a). check to see it works.

d Draw the two eigenvectors in the plane, and describe the linear transformation $\mathbb{A} : \mathbb{R}^2 \to \mathbb{R}^4$, $\begin{pmatrix} x \\ y \end{pmatrix} \mapsto A \begin{pmatrix} x \\ y \end{pmatrix}$ in terms of them.

e Since A is symmetric, the eigenvalues should be real, and the eigenvectors should be $\perp$. Check this.

f Find a matrix Q with $Q^{-1} A Q = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$. [Hint: the rows (or columns, I'm not sure which) of Q should be the eigenvectors of A.] Why? [Hint:

write $Q = \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix}$; then $AQ = Q \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = (\lambda \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \lambda_2 \begin{pmatrix} x_2 \\ y_2 \end{pmatrix})$.]

2 Write out the characteristic polynomial $P_D(t)$ of the diagonal matrix $D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ $(\lambda_i \in \mathbb{R})$. check $P_D(D) = 0$. Do you recognize the coefficients of P_D? What are the roots of P_D? When are they real?

3 Same as 2 but for a general 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

*4 Show that the coefficients of the polynomial P_A are invariant under conjugation, i.e. if $B = C^{-1}AC$, then $P_D = P_A$. Hint: What is $det(C^{-1}(A - tI)C)$?

5 Show that if A is conjugate to a diagonal matrix, (i.e. if $A = B^{-1}DB$ with D diagonal) then A satisfies its characteristics polynomial.

6 Show that $\{A | A \text{satisfies its} CP\}$ is a **closed** subset of $\mathbb{R}^{n^2}$. (This means that any matrix that is the limit of a sequence of "diagonalizable" matrices satisfies its CP.)

7 Show that the matrix $\begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix}$ has only one eigenvector. $(r \neq 0)$ What are the eigenvalues? Show that this matrix is the limit of a sequence of diagonalizable matrices: What are the eigenvalues of the matrix $\begin{pmatrix} 1 & r \\ 0 & 1+\varepsilon \end{pmatrix}$ $(r, \varepsilon > 0)$? This matrix has 2 eigenvectors; what happens to them as $\varepsilon \to 0$?

8 Let A be an $n \times n$ matrix. For a vector $\overline{x} \neq \overline{o}$ in $\mathbb{R}^n$, let $F(\overline{x}) = \frac{\|A\overline{x}\|}{\|\overline{x}\|}$ (so F measures the factor by which A stretches $\overline{x}$.)

a Show that F is bounded on the sphere $\|\overline{x}\| = 1$.

b Show that F is bounded on $\{\overline{x} | \overline{x} \neq \overline{o}\} \subset \mathbb{R}^n$ (Hint: What is $F(\lambda\overline{x})$ if $\lambda > 0$)?

c Give a necessary and sufficient condition on A so that there is an $\epsilon > 0$ with $F(\overline{x}) \geq \varepsilon \ \forall \overline{x} \neq \overline{o}$.

9 A matrix A is **positive definite** if

$$< A\overline{x}, \overline{x} > \geq 0$$

with equality if and only if $\overline{x} = 0$.

Suppose A is a **symmetric** $n \times n$ matrix. This implies that $\mathbb{R}^n$ has an orthonormal basis of eigenvectors $\{\overline{v}_i, \ldots \overline{v}_n\}$ for A, so that

$$< \overline{v}_i, \overline{v}_j > = \begin{cases} 1 \ if \ i = j \\ 0 \ if \ i \neq j \end{cases} \qquad (orthonormal)$$

$$A\overline{v}_i = \lambda_i \overline{v}_i \forall_i (eigenvectors)$$

and any $\overline{v} \in \mathbb{R}^n$ can be written

$$\overline{v} = \sum_{i=1} a_i \overline{v}_i \qquad (basis)$$

More over the eigenvalues of a symmetric matrix are always real.

a If A is symmetric, show that A is positive definite if and only if all the eigenvalues of A are positive.

b Show that (if A is **not** symmetric) it is possible for A to have only real, positive eigenvalues, but not be positive definite. Hint: Consider $A = \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix}$, $|r|$ large. What are the roots of the characteristic polynomial? Can you find $\overline{x}$ with $< \overline{x}, A\overline{x} >< 0$? Exactly where does the proof from (a) fail if A is not symmetric?

c The matrix $\begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix}$ is not conjugate to a diagonal matrix. (Maybe that is "the problem" with it.) Can you find a matric B so that

(i) B is conjugate to a daigonal matrix

(ii) B has positive, distinct, real eigenvalues.

(iii) B is not positive definite.

Hint: Consider the matrix $\begin{pmatrix} 1 & r \\ 0 & 1+\varepsilon \end{pmatrix}$ and use continuity.

Theorem 4.