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Advanced Lecture Course
Lecture 4: Variations and open questions

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There are natural open questions arising from what we have seen in the previous lectures:

- One can wonder what happens if $\mathbb{Q}_p$ is replaced by some finite extension $K/\mathbb{Q}_p$, i.e. if we consider $G_K := \mathrm{Gal}(\overline{\mathbb{Q}_p}/K)$-representations on the Galois side and $GL_2(K)$-representations on what we have called the GL_n side in the first lecture;
- One can also ask what happens for n-dimensional representations with n arbitrary. Not much is known for $n \neq 2$: some partial results have been obtained by people like Herzig [H], Ollivier [O1] [O2], Ollivier-Sécherre [OS], Schraen [Sch], Vignéras [V].

These questions are very difficult: for example, one can naturally define (φ, Γ)-modules for any integer n and any finite extension K of $\mathbb{Q}_p$, but Colmez' functor only gives $B_2(\mathbb{Q}_p)$-representations.

The aim of this talk is to present more naive open questions that arise on the Galois side. More precisely, we focus on the two following topics (inspired by a work in progress of Fontaine-Mézard):

- the reduction modulo p of crystalline representations;
- a generalization of (φ, Γ)-modules.

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1 Modulo p reduction for crystalline representations

Let E be a finite extension of $\mathbb{Q}_p$ (with $p \neq 2$) with ring of integers $\mathcal{O}_E$, maximal ideal $\mathfrak{m}_E$, uniformizer ϖ_E and residue field $k_E := \mathcal{O}_E/\mathfrak{m}_E$. For any integer $k \geq 2$ and any $a_p \in \mathfrak{m}_E$, we define a filtered φ-module $D_{k,a_p} := Ee_1 \oplus Ee_2$ with Frobenius map given by $Mat(\varphi) = \begin{pmatrix} 0 & -1 \\ p^{k-1} & a_p \end{pmatrix}$ and filtration defined by

$$Fil^i D_{k,a_p} := \begin{cases} D_{k,a_p} & \text{if } i \leq 0 \ ; \\ Ee_1 & \text{if } 1 \leq i \leq k-1 \ ; \\ \{0\} & \text{if } i \geq k \ . \end{cases}$$

By Colmez-Fontaine Theorem (Lecture 2), this filtered φ-module is attached to a crystalline representation $V_{k,a_p} : G_{\mathbb{Q}_p} \to GL_2(E)$ such that

$$D_{cris}(V^*_{k,a_p}) = D_{k,a_p} \ .$$

Considering Colmez-Fontaine Theorem, a first naive question could be the following one:

Question 1. *Can we compute the admissible filtered (φ, N)-modules for all $n \geq 2$?*

Some computations due to Dousmanis [D], Ghate [GM] and Imai [I] give a complete description of admissible filtered (φ, N)-modules for $n = 2$. For $n \geq 3$, this is still an open problem.

Now let T be a $G_{\mathbb{Q}_p}$-stable lattice of V_{k,a_p} and $\overline{V}_{k,a_p}$ be the semi-simplification of $T/\varpi_E T$. It is known that $\overline{V}_{k,a_p}$ only depends on V_{k,a_p} (and not on the choice of T), so that we would like to describe $\overline{V}_{k,a_p}$ in terms of k and a_p. We only have partial results when $k \geq 2p+1$ in the following cases:

- if $v_p(a_p) > \lfloor \frac{k-2}{p-1} \rfloor$ (very big), then $\overline{V}_{k,a_p} = \mathrm{ind}(\omega_2^{k-1})$;
- if $0 < v_p(a_p) < 1$ (very small), then $\overline{V}_{k,a_p}$ can be described with a parameter $t \in \{1, \dots, p-1\}$ congruent to $k-1$ modulo $p-1$. The description depends on whether $p-1$ divides $k-3$ or not.

For an explicit description of all the cases where $\overline{V}_{k,a_p}$ is known, we refer to [Be1, Theorem 5.2.1].

We also have the following general result, due to Berger-Breuil [BeBr], that relates what is above to the Langlands correspondence:

Theorem 1. *If V is an absolutely irreducible 2-dimensional E-linear representation of $G_{\mathbb{Q}_p}$, then the semi-simplification $\overline{V}$ corresponds by the local modulo p Langlands correspondence to $\overline{\Pi}(V)$.*

Conjecture 1 (Buzzard-Gee, [BGe]). *If $p \neq 2$, if k is even and if $\overline{V}_{k,a_p}$ is reducible, then $v_p(a_p)$ is an integer.*

$\rightsquigarrow$ The strategy is then to find an algorithm able to compute $\overline{V}_{k,a_p}$ for n large enough starting from the data $(k, a_p \bmod \varpi_E^n)$. Two useful tools can be used to reach this goal as they give a way to build lattices and to make modulo p reductions:

- Wach modules [Be2];
- Breuil-Kisin modules [K].

2 Generalization of (φ, Γ)-modules

Let k be a perfect field of characteristic $p > 0$ and let $\sigma : x \mapsto x^p$ be its Frobenius map. Denote by $W = W(k)$ the ring of Witt vectors with coefficients in k and by $K_0 := W[\frac{1}{p}]$ its fraction field. We can extend the Frobenius σ to W and then to K_0.

Let K be a totally ramified extension of K_0 of finite degree e: we can then write $K = K_0(\pi_0)$. Fix some algebraic closure $\overline{K}$ of K, denote by $\mathfrak{m}_{\overline{K}}$ its maximal ideal and by $q_0 \in W[X]$ the minimal polynomial of π_0 over K_0: it's an Eisenstein polynomial satisfying $q_0(\pi_0) = 0$.

We set the following definition: a **φ-data** is a data $(\mathcal{F}, (\pi_n)_{n\in\mathbb{N}})$ with:

- $\mathcal{F} = \sum_{i\geq 0} a_i X^i \in W[[X]]$ such that $\mathcal{F}(X) \equiv X^p\ [p]$;
- $(\pi_n)_{n\in\mathbb{N}}$ is a compatible system of elements of $\mathfrak{m}_{\overline{K}}$ such that:

$$\forall\ n \geq 1,\ \sum_{i\geq 0} \sigma^{-n}(a_i)\pi_n^i = \pi_{n-1}\ .$$

Let $(\mathcal{F},\ (\pi_n)_{n\in\mathbb{N}})$ be a φ-data. For any $n \geq 1$, set $K_n := K[\pi_n]$ and $K_\infty := \bigcup_{n\geq 1} K_n$. Also set $K_{cyc} := K(\mu_{p^\infty})$ and $L := K_\infty K_{cyc}$ the composite extension.
A naive question that naturally arises is the following:

Question 2. *For which $\mathcal{F}$ is the extension L/K a Galois extension?*

We don't know so far if there are many such datas or not. We only know two examples of cases where it is actually Galois:

1^{st} **example:** **The cyclotomic tower:**
Let $\pi_0 := \zeta_0 - 1$ with ζ_0 a primitive p-root of unity and $K := K_0(\zeta_0)$.
Consider $\mathcal{F}(X) := (X + 1)^p - 1$ and π_n given by $\zeta_n = 1 + \pi_n$ where ζ_n is a p-th root of ζ_{n-1}. Then $(\mathcal{F}, (\pi_n)_{n\in\mathbb{N}})$ is a φ-data and the fields K_n that it defines are precisely the fields of the cyclotomic tower of K.

2^{nd} **example:** **The Kummer extension:**
It does correspond to the following φ-data: $\mathcal{F}(X) := X^p$ and $\pi_n^p = \pi_{n-1}$.

Now return to the general setting and consider a φ-data $(\mathcal{F},\ (\pi_n)_{n\in\mathbb{N}})$: since $\mathcal{F}(X) \equiv X^p\ [p]$, we have $\pi_n^p \equiv \pi_{n-1}$ so that $(\pi_n)_{n\in\mathbb{N}}$ defines an element of

$\tilde{\mathbf{E}}$, which is in fact in $\tilde{\mathbf{E}}^+$ as $val(\pi_0) \geq 0$ by definition. We can then consider $[\pi] = [(\pi_n)_{n\in\mathbb{N}}] \in \tilde{\mathbf{A}}^+ := W(\tilde{\mathbf{E}}^+)$, and we easily have the following result:

Lemma 1. *There exists a unique continuous injective morphism of W-algebras $W[[u]] \to \tilde{\mathbf{A}}^+$ commuting with the Frobenius maps and such that u has image π through the composite map $W[[u]] \longrightarrow \tilde{\mathbf{A}}^+ \twoheadrightarrow \tilde{\mathbf{E}}^+$.*

Lemma 1 then allows us to see $W[[u]]$ as a subring of $\tilde{\mathbf{A}}^+$. Remember that we saw in Lecture 2 that there is a map $\theta : \tilde{\mathbf{A}}^+ \to \mathcal{O}_{\mathbb{C}_p}$ that extends to $\tilde{\mathbf{B}}^+ \to \mathbb{C}_p$. If we let $H := \mathrm{Gal}(\overline{K}/K_\infty)$, we then can prove the following theorem:

Theorem 2. *We have the following equivalences of categories:*

$$\{\mathbb{F}_p\text{-representations of } H\} \leftrightarrow \left\{\varphi\text{-modules over } \mathbf{E}_{\mathbb{Q}_p}\right\} \ ;$$

$$\{\text{free } \mathbb{Z}_p\text{-representations of } H\} \leftrightarrow \left\{\varphi\text{-modules over } \mathbf{A}_{\mathbb{Q}_p}\right\} \ ;$$

$$\{\mathbb{Q}_p\text{-representations of } H\} \leftrightarrow \left\{\text{étale } \varphi\text{-modules over } \mathbf{B}_{\mathbb{Q}_p}\right\} \ .$$

It could be interesting to compare these categories for different φ or different H. In the case of a Galois extension, we can also define a theory of (φ, Γ)-modules based on these constructions, but this is a whole new story...

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