Ariane Mézard* Ramla Abdellatif†

Advanced Lecture Course
Lecture 3: Modulo p Galois representations and beyond

2010 Program for Women and Mathematics

Table des matières

We have seen in the previous lecture that Colmez-Fontaine equivalence of categories between crystalline representations and admissible filtered φ-modules gives an explicit classification of crystalline representations of $G_{\mathbb{Q}_p}$, and then an explicit p-adic local Langlands correspondence in dimension 2.
To take into account all the p-adic representations of $G_{\mathbb{Q}_p}$, we have to consider a new family of objects called (φ, Γ)-modules. We will only describe here the ideas involved in this theory and its main steps. For further details and proofs, we refer to [Be, Section 9], [BeBr, Section 3] and to M.-F. Vignéras lectures during the next week.

*Laboratoire de Mathématiques - Université de Versailles Saint-Quentin , 78 035 Versailles; Ariane.Mezard@math.uvsq.fr
†Laboratoire de Mathématiques - Université Paris-Sud XI , 91 405 Orsay; Ramla.Abdellatif@math.u-psud.fr

1 What you need to know / Prerequisites

- The classification of 2-dimensional modulo p Galois representations

$$\rho : G_{\mathbb{Q}_p} \longrightarrow GL_2(\overline{\mathbb{F}}_p)$$

will be recalled in the next section and is constructed in Advanced Homework Session 1.

- We have the following result, which is proved in Advanced Homework Sessions 3 and 4:

Theorem 1. *Assume that* $\mathbf{E}$ *is endowed with the discrete topology.*

1. $H^1(H_{\mathbb{Q}_p}, GL_d(\mathbf{E})) = \{1\}$ *for any integer* $d \geq 1$.
2. $H^1(H_{\mathbb{Q}_p}, \mathbf{E}) = \{0\}$.

Basically, this theorem says that any free $\mathbf{E}$-module of rank d endowed with a semi-linear action of $H_{\mathbb{Q}_p}$ is isomorphic to $\mathbf{E}^d$. We refer to Advanced Homework Session 3 for more details.

2 Modulo p representations of $G_{\mathbb{Q}_p}$

2.1 The 2-dimensional case

As we mentionned it in the previous section, we have a complete classification of absolutely irreducible representations of $G_{\mathbb{Q}_p}$ on finite fields (Advanced Homework Session 1):

Theorem 2. *Let* E *be a finite extension of* $\mathbb{Q}_p$ *with residue field* k_E.
Any 2*-dimensional* k_E*-linear absolutely irreducible representation of* $G_{\mathbb{Q}_p}$ *is isomorphic to* $\rho(r, \chi) = ind(\omega_2^{r+1}) \otimes \chi$ *for some integer* $0 \leq r \leq p-1$ *and some character* $\chi : G_{\mathbb{Q}_p} \to k_E^{\times}$.

Here we denote by $\omega_2 : I_{\mathbb{Q}_p} \to \mathbb{F}_{p^2}^{\times}$ the character sending g to $\dfrac{g(p^{\frac{1}{p^2-1}})}{p^{\frac{1}{p^2-1}}}$ and $\text{ind}(\omega_2^r)$ is defined as the unique semi-simple representation ρ of dimension 2 whose determinant is equal to ω_2^r and such that $\rho|_{I_{\mathbb{Q}_p}} = \omega_2^r \oplus \omega_2^{pr}$.
The problem is that this classification offers a too naive point of view to be generalized. To become less naive, we have to introduce a new category of objects called (φ, Γ)-modules.

2.2 (φ, Γ)-modules in characteristic p

First recall that $\tilde{\mathbf{E}}$ is (non-canonically) isomorphic to the algebraic closure of $\mathbf{E}_{\mathbb{Q}_p} := \mathbb{F}_p((\epsilon - 1))$. We denote by $\mathbf{E}$ the separable closure of $\mathbf{E}_{\mathbb{Q}_p}$, which is not equal to its algebraic closure! The following result, which is a particular case of a powerful theorem due to Fontaine-Wintenberger [FW], will be proved in Advanced Homework Session 3:

Theorem 3. *There exists a group isomorphism:*

$$H_{\mathbb{Q}_p} \simeq Gal(\mathbf{E}/\mathbf{E}_{\mathbb{Q}_p}) \ .$$

Denote by $\Gamma = \Gamma_{\mathbb{Q}_p}$ the Galois group of $\mathbb{Q}_{p^\infty}/\mathbb{Q}_p$. A **$(\varphi, \Gamma)$-module over $\mathbf{E}_{\mathbb{Q}_p}$** is a free $\mathbf{E}_{\mathbb{Q}_p}$-module of finite rank d endowed with a semi-linear Frobenius map φ such that $\mathrm{Mat}(\varphi) \in GL_d(\mathbf{E}_{\mathbb{Q}_p})$ and a continuous semi-linear action of Γ that commutes with φ.
We do the two following remarks:

1. First note that the condition on $\mathrm{Mat}(\varphi)$ does not depend on the basis in which the matrix of φ is considered.

2. If we choose a basis e of D, an element $\gamma \in \Gamma$ and set $P := Mat_e(\varphi)$ and $G := Mat_e(\gamma)$, then requiring that ϕ and γ commute as semi-linear operators is equivalent to require that $P\varphi(G) = G\gamma(P)$.

Assuming the fact that $\mathbf{E}^{H_{\mathbb{Q}_p}} = \mathbf{E}_{\mathbb{Q}_p}$ (Advanced Homework Session 3), we can prove the following result:

Proposition 1. *Let W be an $\mathbb{F}_p$-representation of $G_{\mathbb{Q}_p}$ of dimension d.*
Then $D(W) := (\mathbf{E} \otimes_{\mathbb{F}_p} W)^{H_{\mathbb{Q}_p}}$ is a (φ, Γ)-module over $\mathbf{E}_{\mathbb{Q}_p}$ of dimension d such that

$$\mathbf{E} \otimes_{\mathbf{E}_{\mathbb{Q}_p}} D(W) \simeq E \otimes_{\mathbb{F}_p} W \ .$$

In particular, we have

$$W = (\mathbf{E} \otimes_{\mathbf{E}_{\mathbb{Q}_p}} D(W))^{\varphi=1} \ .$$

Démonstration. Let $W : G_{\mathbb{Q}_p} \to GL_d(\mathbb{F}_p)$ be such a representation. Its restriction to $H_{\mathbb{Q}_p}$ defines an element $[W \otimes_{\mathbb{F}_p} \mathbf{E}]$ of $H^1(H_{\mathbb{Q}_p}, GL_d(\mathbf{E}))$, which is trivial by the first point of Theorem 1. This means that $W \otimes_{\mathbb{F}_p} \mathbf{E}$ is isomorphic to $\mathbf{E}^d$ as an $H_{\mathbb{Q}_p}$-representation, so that $D(W) := (\mathbf{E} \otimes_{\mathbb{F}_p} W)^{H_{\mathbb{Q}_p}}$ is an $\mathbf{E}_{\mathbb{Q}_p}$-vector space of dimension d which is stable under φ and Γ. Moreover, we have an isomorphism of $E_{\mathbb{Q}_p}$-vector spaces (but not of (φ, Γ)-modules):

$$D(W) \simeq \mathbf{E}_{\mathbb{Q}_p}^d$$

so that $(\mathbf{E} \otimes_{\mathbf{E}_{\mathbb{Q}_p}} D(W))^{\varphi=1} = W$. □

Proposition 2. *Let D be a (φ, Γ)-module of rank d over $\mathbf{E}_{\mathbb{Q}_p}$.*
Then $W(D) := (\mathbf{E} \otimes_{\mathbf{E}_{\mathbb{Q}_p}} D)^{\varphi=1}$ is an $\mathbb{F}_p$-representation of $G_{\mathbb{Q}_p}$ of dimension d such that

$$\mathbf{E} \otimes_{\mathbb{F}_p} W(D) \simeq \mathbf{E} \otimes_{\mathbf{E}_{\mathbb{Q}_p}} D \ .$$

Démonstration. [Be, Proposition 9.1.5] □

Corollaire 1. *The map $[W \mapsto D(W)]$ defines an equivalence of categories:*

$$\{\mathbb{F}_p\text{-linear representations of } G_{\mathbb{Q}_p}\} \leftrightarrow \{(\varphi, \Gamma)\text{-modules over } \mathbf{E}_{\mathbb{Q}_p}\} \ .$$

Démonstration. [Be, Theorem 9.1.8] □

To finish this section, note that if we forget about the action of Γ, we then get the following result:

Corollaire 2. *There exists an equivalence of categories:*

$$\left\{\mathbb{F}_p\text{-linear representations of } H_{\mathbb{Q}_p}\right\} \leftrightarrow \left\{\varphi\text{-modules over } \mathbf{E}_{\mathbb{Q}_p}\right\} \ .$$

3 Application to a mod p Langlands correspondence for $n = 2$

We just saw how to go from Galois representations to (φ, Γ)-modules. We now introduce an operator ψ on these (φ, Γ)-modules in order to define (as Colmez did) a representation of the Borel subgroup $B_2(\mathbb{Q}_p) \subset GL_2(\mathbb{Q}_p)$.

3.1 The operator ψ

Recall that $\mathbf{E}_{\mathbb{Q}_p} := \mathbb{F}_p((\epsilon - 1))$ is a vector space over $\varphi(\mathbf{E}_{\mathbb{Q}_p}) = \mathbb{F}_p((\varphi(\epsilon - 1)))$ which admits $\{1, \epsilon, \ldots, \epsilon^{p-1}\}$ as a basis. Any $\alpha \in \mathbf{E}_{\mathbb{Q}_p}$ can therefore be uniquely written as

$$\alpha = \sum_{j=0}^{p-1} \epsilon^j \alpha_j$$

with $\alpha_j \in \varphi(\mathbf{E}_{\mathbb{Q}_p})$. We set $\psi(\alpha) := \alpha_0$.
Let now D be a (φ, Γ)-module over $\mathbf{E}_{\mathbb{Q}_p}$: then D admits a basis $(\varphi(e_1), \ldots, \varphi(e_d))$ made from elements of $\varphi(D)$, so that any $x \in D$ can be uniquely written

$$x = \sum_{j=1}^{d} x_j \varphi(e_j)$$

with $x_j \in \mathbf{E}_{\mathbb{Q}_p}$. We set $\psi(x) := \sum_{j=1}^{d} \psi(x_j) e_j$.

Lemma 1. *The map $\psi : D \to D$ defined just above doesn't depend on the choice of the basis $(\varphi(e_1), \ldots, \varphi(e_d))$ and commutes to the action of Γ.*

3.2 Main steps to a mod p Langlands correspondence

The construction of a $GL_2(\mathbb{Q}_p)$-representation starting from a (φ, Γ)-module attached to a $G_{\mathbb{Q}_p}$-representation, known as **Colmez' functor**, splits into three main steps:

1^{st} step: Let D be a (φ, Γ)-module. There exists some ψ-stable lattice N in D, and we let $(\varprojlim_{\psi} D)^b$ be the set of elements $x = (x_n)_{n \in \mathbb{N}} \in D^{\mathbb{N}}$ satisfying the two following conditions:

$$\begin{cases} \forall\, n \in \mathbb{N},\ \psi(x_{n+1}) = x_n \ ; \\ \exists\, k \in \mathbb{N} \mid \forall\, n \in \mathbb{N},\ x_n \in \pi^{-k} N \ . \end{cases}$$

2^{nd} step: We set $D^\sharp := \{x_0,\ x \in (\varprojlim_{\psi} D)^b\}$. One can prove that $D^\sharp$ is stable under ψ and Γ, and that $\psi : D^\sharp \to D^\sharp$ is surjective. We can also build $\varprojlim_{\psi} D^\sharp$ starting from $D^\sharp$ as we built $(\varprojlim_{\psi} D)^b$ starting from D.

3^{rd} step: Let $\chi : \mathbb{Q}^\times \to E^\times$ be a smooth character. We endow $\varprojlim_{\psi} D^\sharp$ with an action of the Borel subgroup $B_2(\mathbb{Q}_p)$ as follows:

$$\begin{cases} \left(\begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix} x\right)_n = \chi^{-1}(t)x_n \ ; \quad \left(\begin{pmatrix} 1 & 0 \\ 0 & p^j \end{pmatrix} x\right)_n = x_{n-j} = \psi^j(x_n) \\ ; \\ \left(\begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix} x\right)_n = \gamma_{a^{-1}}(x_n) \ ; \quad \left(\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} x\right)_n = \pi^{p^n z} x_n \ . \end{cases}$$

Here $x \in \varprojlim_{\psi} D^\sharp$, $a \in \mathbb{Z}_p^\times$ and $z \in \mathbb{Q}_p$. Moreover, we set $\gamma_{a^{-1}}$ the element of Γ such that $\chi_{cyc}(\gamma_{a^{-1}}) = a^{-1}$ (where χ_{cyc} denotes the cyclotomic character).
The key point of this construction is that if D is the (φ, Γ)-module associated to an absolutely irreducible representation of $G_{\mathbb{Q}_p}$, then this representation of $B_2(\mathbb{Q}_p)$ lifts in a unique way to an irreducible representation of $GL_2(\mathbb{Q}_p)$.

4 First steps in characteristic 0

As usual, we want to lift to characteristic 0 what we have done in characteristic p. To do this, we will (as usual) introduce some new rings of period.
Let $\tilde{\mathbf{A}} := W(\tilde{\mathbf{E}})$ be the ring of Witt vectors with coefficients in $\tilde{\mathbf{E}}$. Denote by $\mathbf{A}_{\mathbb{Q}_p}$ the p-adic completion of $\mathbb{Z}_p[[\pi]][\frac{1}{\pi}]$ inside $\tilde{\mathbf{A}}$ and set $\mathbf{B}_{\mathbb{Q}_p} := \mathbf{A}_{\mathbb{Q}_p}[\frac{1}{p}]$: this is a local field with residue field equal to $\mathbf{E}_{\mathbb{Q}_p}$.
Let $\tilde{\mathbf{B}} := \tilde{\mathbf{A}}[\frac{1}{p}]$ and denote by $\mathbf{B}$ the p-adic completion of the maximal unramified extension of $\mathbf{B}_{\mathbb{Q}_p}$ inside $\tilde{\mathbf{B}}$. Finally set $\mathbf{A} := \tilde{\mathbf{A}} \cap \mathbf{B}$.
We then set the following definitions:

- A (φ, Γ)**-module over** $\mathbf{A}_{\mathbb{Q}_p}$ is a free $\mathbf{A}_{\mathbb{Q}_p}$-module D of finite rank d equipped with a semi-linear Frobenius φ such that $Mat(\varphi) \in GL_d(\mathbf{A}_{\mathbb{Q}_p})$ and a continuous semi-linear action of Γ which commutes to φ.
- A (φ, Γ)**-module over** $\mathbf{B}_{\mathbb{Q}_p}$ is a free $\mathbf{B}_{\mathbb{Q}_p}$-module D of finite rank d equipped with a semi-linear Frobenius φ such that $Mat(\varphi) \in GL_d(\mathbf{B}_{\mathbb{Q}_p})$ and a continuous semi-linear action of Γ which commutes to φ.
- We say that a (φ, Γ)-module over $\mathbf{B}_{\mathbb{Q}_p}$ is **étale** if there exists a basis e of D such that $Mat_e(\varphi) \in GL_d(\mathbf{A}_{\mathbb{Q}_p})$.

As in characteristic p, we have $H^1(H_{\mathbb{Q}_p}, GL_d(\mathbf{A})) = \{1\}$ if $\mathbf{A}$ is endowed with the p-adic topology (Advanced Homework Session 4). This leads to the following results:

Lemma 2. *Let T be a free $\mathbb{Z}_p$-module of finite rank d endowed with a continuous action of $G_{\mathbb{Q}_p}$. Then $D(T) := (\mathbf{A} \otimes_{\mathbb{Z}_p} T)^{H_{\mathbb{Q}_p}}$ is a (φ, Γ)-module of rank d over $\mathbf{A}_{\mathbb{Q}_p}$ and it satisfies:*

$$\mathbf{A} \otimes_{\mathbf{A}_{\mathbb{Q}_p}} D(T) \simeq \mathbf{A} \otimes_{\mathbb{Z}_p} T \ .$$

In particular, we have:

$$(\mathbf{A} \otimes_{\mathbf{A}_{\mathbb{Q}_p}} D(T))^{\varphi=1} = T \ .$$

Lemma 3. *Let V be a free $\mathbb{Q}_p$-module of finite rank d endowed with a continuous action of $G_{\mathbb{Q}_p}$. Then $D(V) := (\mathbf{B} \otimes_{\mathbb{Q}_p} V)^{H_{\mathbb{Q}_p}}$ is an étale (φ, Γ)-module of rank d over $\mathbf{B}_{\mathbb{Q}_p}$ and it satisfies:*

$$\mathbf{B} \otimes_{\mathbf{B}_{\mathbb{Q}_p}} D(V) \simeq \mathbf{B} \otimes_{\mathbb{Q}_p} V$$

In particular, we have

$$(\mathbf{B} \otimes_{\mathbf{B}_{\mathbb{Q}_p}} D(V))^{\varphi=1} = V$$

Theorem 4. *The functor $D(.)$ defines equivalences of categories:*

$$\left\{\text{free } \mathbb{Z}_p\text{-representations of } G_{\mathbb{Q}_p}\right\} \leftrightarrow \left\{(\varphi, \Gamma)\text{-modules over } \mathbf{A}_{\mathbb{Q}_p}\right\} \ .$$

$$\left\{\mathbb{Q}_p\text{-linear representations of } G_{\mathbb{Q}_p}\right\} \leftrightarrow \left\{\text{étale } (\varphi, \Gamma)\text{-modules over } \mathbf{B}_{\mathbb{Q}_p}\right\} \ .$$

Once again, if we forget about the Γ-action, we have the following result:

Corollaire 3. *We have the following equivalences of categories:*

$$\left\{\text{free } \mathbb{Z}_p\text{-representations of } H_{\mathbb{Q}_p}\right\} \leftrightarrow \left\{\varphi\text{-modules over } \mathbf{A}_{\mathbb{Q}_p}\right\} \ ;$$

$$\left\{\mathbb{Q}_p\text{-linear representations of } H_{\mathbb{Q}_p}\right\} \leftrightarrow \left\{\text{étale } \varphi\text{-modules over } \mathbf{B}_{\mathbb{Q}_p}\right\} \ .$$

It is also possible to define an operator ψ and to make the three-steps construction that has been seen in characteristic p, but it is harder to go from $B_2(\mathbb{Q}_p)$-representations to $GL_2(\mathbb{Q}_p)$-representations in the characteristic 0 setting. In fact, one can prove that it works for some big enough family of p-adic representations so that we can conclude that it works for any p-adic representation by density. For further details, we refer to [C, Section II.3.2].

Références

[Be] L. Berger, *Partial notes for the course "Galois representations and (φ, Γ)-modules"*, Galois Trimester 2010 at the IHP.

[BeBr] L. Berger, Ch. Breuil, *Towards a Langlands correspondence*, Notes of a course given at Hangzhou C.M.S (2004).

[C] P. Colmez, *Construction de représentations galoisiennes de $GL_2(\mathbb{Q}_p)$*, cours à l'X 2008-2009, disponible sur sa page web.

[FW] J.-M. Fontaine, J.-P. Wintenberger, *Le « corps des normes » de certaines extensions algébriques de corps locaux*, C.R. Acad. Sc. Paris Sér. A-B 288 (1979), 6, A367–A370.