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Advanced Lecture Course
Lecture 2: Crystalline representations

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In the previous lecture, we have seen that the main difficulty to build a p-adic Langlands correspondence is that the category of p-adic continuous finite-dimensional representations of $G_{\mathbb{Q}_p}$ is far too big. An idea of J.-M. Fontaine was to introduce some subcategories of p-adic representations which could be easier to manipulate. A very interesting one is the subcategory of crystalline representations, as it gives rise to very explicit formulas leading to an explicit correspondence.

The aim of this lecture is to define what crystalline representations are and to classify them. For further details and proofs, we refer to [BeBr, Chapter 2].

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1 What you need to know / Prerequisites

Here are some tools and results that will be used during this lecture and will be seen in E. Mantovan's lectures or during the homework sessions of R. Abdellatif and L. Peskin:

- $\mathbb{C}_p := \widehat{\overline{\mathbb{Q}}_p}$ is an algebraically closed field whose ring of integers is denoted by $\mathcal{O}_{\mathbb{C}_p}$.
- If A is a perfect $\mathbb{F}_p$-algebra, what means that $[x \mapsto x^p]$ is an automorphism, we can define its ring of Witt vectors $W(A)$ [B, Chapitre IX, Section 1.4]. For example, if $A = \overline{\mathbb{F}}_p$, then $W(A) = \mathbb{Z}_p$. A Witt vectors ring is a characteristic 0 ring and it has nice functorial properties:
  - any $x \in A$ can be canonically lifted as $[x] \in W(A)$. This lift is called the **Teichmüller lift**.
  - Similarly, any ring homomorphism $f : A \to A$ can be lifted to a ring homomorphism $W(A) \to W(A)$.
- We will use modules, tensor product $\otimes$ and exterior product $\wedge$.
- Using Galois Theory, one can get an explicit description of Galois representations of dimension 1 (= Galois characters): they all can be written as $\omega^n \mu_\lambda$, where ω is the p-adic cyclotomic character, n some integer and μ_λ the unramified character associated to λ.
- There is an explicit description of crystalline characters (Advanced Homework Session 2).

2 Introduction to Fontaine's theory

Let $\rho_p : G_{\mathbb{Q}_p} \to GL(V)$ be some p-adic representation with V an E-vector space (for E a finite extension of $\mathbb{Q}_p$). Assume that $\mathbb{B}$ is a topological $\mathbb{Q}_p$-algebra endowed with a continuous linear action of $G_{\mathbb{Q}_p}$.

We say that V is **$\mathbb{B}$-admissible** if we have an isomorphism of $\mathbb{B}[\mathbb{Q}_p]$-modules:

$$\mathbb{B} \otimes_{\mathbb{Q}_p} V \simeq \mathbb{B}^{\dim_{\mathbb{Q}_p} V} .$$

Such a V shouldn' t be too much complicated to understand once we know enough properties of $\mathbb{B}$. Moreover, the $\mathbb{B}$-admissible representations are a subcategory of the p-adic representations we are studying, and if $\mathbb{B}$ is equipped with some extra structure which commutes with the action of $G_{\mathbb{Q}_p}$, then $D_{\mathbb{B}}(V) := (\mathbb{B} \otimes_{\mathbb{Q}_p} V)^{G_{\mathbb{Q}_p}}$ is endowed with the same structure. This structure will then provide some invariants which will be used to classify the $\mathbb{B}$-admissible representations. For example, we will define in this lecture a ring $\mathbf{B}_{cris}$ which will be endowed with two extra structures: a filtration and a Frobenius map.

3 Some period rings

3.1 $\tilde{\mathbf{E}}$, $\tilde{\mathbf{E}}^+$

Let $\mathbb{C}_p$ be the completion of $\overline{\mathbb{Q}}_p$ for the p-adic topology and $\mathcal{O}_{\mathbb{C}_p}$ be its ring of integers. We set:

$$\begin{cases} \tilde{\mathbf{E}} := \varprojlim\limits_{x \mapsto x^p} \mathbb{C}_p = \{x = (x^{(n)})_{n \in \mathbb{N}},\ x^{(n)} \in \mathbb{C}_p \mid (x^{(n+1)})^p = x^{(n)}\} \ ; \\ \\ \tilde{\mathbf{E}}^+ := \{x \in \tilde{\mathbf{E}} \mid x^{(0)} \in \mathcal{O}_{\mathbb{C}_p}\} \ . \end{cases}$$

We can put a ring structure on these sets using the following rules: if x and y are in $\tilde{\mathbf{E}}$, we define their sum $x+y$ and their product xy by

$$\begin{cases} (x+y)^{(n)} := \lim\limits_{j \to \infty} (x^{(n+j)} + y^{(n+j)})^{p^j} \ ; \\ \\ (xy)^{(n)} := x^{(n)} y^{(n)} \ . \end{cases}$$

This makes $\tilde{\mathbf{E}}$ into a field of characteristic p. Moreover, we can define a valuation on $\tilde{\mathbf{E}}$ by setting

$$v_{\tilde{\mathbf{E}}}(x) := v_p(x^{(0)}) \ .$$

We then get the following result due to Fontaine [F]:

Theorem 1. *1. $\tilde{\mathbf{E}}^+$ is the ring of integers of $\tilde{\mathbf{E}}$ for $v_{\tilde{\mathbf{E}}}$.*

2. $\tilde{\mathbf{E}}$ is complete for the topology defined by $v_{\tilde{\mathbf{E}}}$.

Furthermore, the ring $\tilde{\mathbf{E}}^+$ contains in particular the two following elements:

$$\begin{cases} \epsilon := (1, \zeta_p, \zeta_{p^2}, \ldots) \text{ where } \zeta_{p^n} \text{ is a primitive } p^n\text{-th root of } 1 \ ; \\ \\ \overline{\pi} := \epsilon - 1 \ . \end{cases}$$

Note that we have $\tilde{\mathbf{E}} = \tilde{\mathbf{E}}^+[\frac{1}{\overline{\pi}}]$ and that $\mathbb{F}_p((\overline{\pi}))$ is contained in $\tilde{\mathbf{E}}$. Actually, $\tilde{\mathbf{E}}$ is isomorphic (in a non-canonical way) to the algebraic closure of $\mathbb{F}_p((\overline{\pi}))$ (Advanced Homework Session 3).

The field $\tilde{\mathbf{E}}$ is equipped with a map $\varphi : \tilde{\mathbf{E}} \longrightarrow \tilde{\mathbf{E}}$ sending x to x^p: it is called the **Frobenius map** on $\tilde{\mathbf{E}}$ and is a field automorphism, so that $\tilde{\mathbf{E}}$ is a perfect field.

Finally, as it is recalled in E. Mantovan lectures, $\mathbb{C}_p$ is endowed with a continuous action of $G_{\mathbb{Q}_p}$ which preserves the property of being a p^{th}-root. This naturally defines a continuous action of $G_{\mathbb{Q}_p}$ on $\tilde{\mathbf{E}}$.

3.2 $\tilde{\mathbf{A}}^+$, $\tilde{\mathbf{B}}^+$

As $\tilde{\mathbf{E}}^+$ is a perfect $\mathbb{F}_p$-algebra, we can consider the ring $\tilde{\mathbf{A}}^+ := W(\tilde{\mathbf{E}}^+)$ of its Witt vectors. We let $\tilde{\mathbf{B}}^+ := \tilde{\mathbf{A}}^+[\frac{1}{p}]$, so that we also have

$$\tilde{\mathbf{B}}^+ = \{ \sum_{k \gg -\infty} p^k [x_k],\ x_k \in \tilde{\mathbf{E}}^+ \}$$

where $[.] : \tilde{\mathbf{E}}^+ \longrightarrow \tilde{\mathbf{A}}^+$ denotes the Teichmüller lifting map. In particular, $\tilde{\mathbf{A}}^+$ contains the following elements:

$$\begin{cases} \pi := [\epsilon] - 1 \ ; \\ \pi_1 := [\epsilon^{\frac{1}{p}}] - 1 \ ; \\ \omega := \pi/\pi_1 \ . \end{cases}$$

Note also that for any $k \in \mathbb{N}$, we have $\overline{\pi}^k \in \tilde{\mathbf{E}}^+$ so that $[\overline{\pi}^k]$ is an element of $\tilde{\mathbf{A}}^+$. The topology on $\tilde{\mathbf{A}}^+$ is then defined by taking as a family of neighborhoods of 0 the following collection of open sets: $\{([\overline{\pi}^k], p^n)\tilde{\mathbf{A}}^+;\ k, n \in \mathbb{N}\}$.

We can moreover endow $\tilde{\mathbf{B}}^+$ with a natural map $\theta : \tilde{\mathbf{B}}^+ \longrightarrow \mathbb{C}_p$ defined as follows:

$$\theta(\sum_{k \gg -\infty} p^k [x_k]) := \sum_{k \gg -\infty} p^k x_k^{(0)} \ .$$

Recall also that the properties of the Witt vectors construction imply that the Frobenius map $\varphi : \tilde{\mathbf{E}}^+ \to \tilde{\mathbf{E}}^+$ can be lifted to a Frobenius map $\varphi : \tilde{\mathbf{A}}^+ \to \tilde{\mathbf{A}}^+$ and then (by inverting p) to a Frobenius map $\varphi : \tilde{\mathbf{B}}^+ \to \tilde{\mathbf{B}}^+$. One can check that we have the following explicit formula:

$$\varphi(\sum_{k \gg -\infty} p^k [x_k]) = \sum_{k \gg -\infty} p^k [x_k^p]$$

so that it is not very hard to see that this map is a bijection. Finally, the action of $G_{\mathbb{Q}_p}$ on $\tilde{\mathbf{E}}^+$ lifts to an action on $\tilde{\mathbf{A}}^+$, and then on $\tilde{\mathbf{B}}^+$.

4 Crystalline representations

4.1 The ring $\mathbf{B}_{cris}$

We set

$$\mathbf{B}_{cris}^+ := \{\sum_{n \geq 0} a_n \frac{\omega^n}{n!},\ a_n \in \tilde{\mathbf{B}}^+ \mid \lim_{n \to \infty} a_n = 0\} \ .$$

The series $-\sum_{n \geq 1} \frac{(1 - [\epsilon])^n}{n}$ does converge in $\mathbf{B}_{cris}^+$ to an element denoted by t. This should be remembered as "$\log([\epsilon])$". We then set $\mathbf{B}_{cris} := \mathbf{B}_{cris}^+[\frac{1}{t}]$. This ring is endowed with a filtration and by a continuous Frobenius map φ extending the Frobenius on $\tilde{\mathbf{B}}^+$. For example, one can check that we have $\varphi t = pt$. As $G_{\mathbb{Q}_p}$ acts continuously and linearly on $\mathbf{B}_{cris}$, we have all the conditions required in the section 2 to define the notion of $\mathbf{B}_{cris}$-admissible representations.

4.2 Crystalline representations

We say that a representation V of $G_{\mathbb{Q}_p}$ is **crystalline** if it is $\mathbf{B}_{cris}$-admissible. This is equivalent to require that the $\mathbb{Q}_p$-vector space

$$D_{cris}(V) := (\mathbf{B}_{cris} \otimes_{\mathbb{Q}_p} V)^{G_{\mathbb{Q}_p}}$$

has dimension $\dim_{\mathbb{Q}_p} V$.
Note that $D_{cris}(V)$ inherits a filtration and a Frobenius from those of $\mathbf{B}_{cris}$. For some examples, we refer to Advanced Homework Session 2.

5 Colmez-Fontaine Theorem

5.1 Filtered φ-modules

A **filtered φ-module** is a finite-dimensional $\mathbb{Q}_p$-vector space endowed with:

- a semi-linear injective Frobenius map $\varphi : D \to D$;
- a decreasing filtration $(Fil^i(D))_{i\in\mathbb{Z}}$ by $\mathbb{Q}_p$-vector spaces such that:
  - $Fil^i D = D$ for $i << 0$;
  - $Fil^i D = \{0\}$ for $i >> 0$.

Let D be a filtered φ-module of dimension d. Then $\bigotimes_{\mathbb{Q}_p}^{d} D$ is also a filtered φ-module for the Frobenius given by $\otimes^d \varphi$ and for the filtration defined by:

$$\forall\ i \in \mathbb{Z},\ Fil^i(\bigotimes^{d} D) := \sum_{i_1+\ldots+i_d=i} Fil^{i_1} D \otimes Fil^{i_2} D \otimes \ldots \otimes Fil^{i_d} D\ .$$

We then can also define a structure of filtered φ-module on $\bigwedge^{d} D := Sym(\bigotimes^{d} D)$, which is a $\mathbb{Q}_p$-vector space of dimension 1. In particular, there exists a unique $i_0 \in \mathbb{Z}$ such that

$$Fil^i(\bigwedge^{d} D) = \begin{cases} \bigwedge^{d} D \text{ if } i \leq i_0\ ; \\ \{0\} \text{ if } i > i_0\ ; \end{cases}$$

and we can write $\bigwedge^{d} D = \mathbb{Q}_p e_0$ with $\varphi e_0 = \lambda_0 e_0$.
We then define $t_N(D) := i_0$, $t_H(D) := v_p(\lambda_0)$ and set the following defintion: we say that D is an **admissible** filtered φ-module if $t_N(D) = t_H(D)$ and if for any sub-φ-module D' of D, we have $t_N(D') \geq t_H(D')$.

5.2 Colmez-Fontaine Theorem

The following result can be proved quite easily:

Lemma 1. *If V is a crystalline p-adic representation of $G_{\mathbb{Q}_p}$, then $D_{cris}(V)$ is an admissible filtered φ-module.*

If we want to go in the reverse direction, it is much far harder. For filtered modules of dimension 2, we have the following theorem due to Colmez-Fontaine [CF]:

Theorem 2 (Colmez-Fontaine)**.** *If V is a p-adic representation of dimension 2, then V is crystalline if and only if $D_{cris}(V)$ is admissible as filtered φ-module.*

In Advanced Homework Session 2, there is a proof of the classification of admissible irreducible filtered φ-modules of dimension 2: they can all be written

$D_{k,a_p} = \overline{\mathbb{Q}_p}e_1 \oplus \overline{\mathbb{Q}_p}e_2$ with $k \geq 2$ being an integer, $a_p \neq \pm(p^{\frac{k}{2}} + p^{\frac{k}{2}-1})$ an element of $\mathfrak{m}_E$, $Mat(\varphi) = \begin{pmatrix} 0 & -1 \\ p^{k-1} & a_p \end{pmatrix}$ and

$$Fil^i D_{k,a_p} := \begin{cases} D_{k,a_p} & \text{if } i \leq 0 \ ; \\ Ee_1 & \text{if } 1 \leq i \leq k-1 \ ; \\ \{0\} & \text{if } i \geq k \ . \end{cases}$$

The associated crystalline representation is denoted by V_{k,a_p} and corresponds to the following representation of $GL_2(\mathbb{Q}_p)$ (that should be studied in M.-F. Vignéras lectures):

$$\Pi_{k,a_p,\chi} := (\underline{Sym}^{k-2}(E^2) \otimes \mathrm{Ind}_{B_2(\mathbb{Q}_p)}^{GL_2(\mathbb{Q}_p)}(\mu_{\lambda_1} \otimes \mu_{p\lambda_2^{-1}})) \otimes (\chi \circ \det)$$

where λ_1 and λ_2 are the roots of $X^2 - a_pX + p^{k-1} = 0$.

$\rightsquigarrow$ As we claimed at the beginning of this lecture, we have explicit formulas that naturally define a Langlands correspondence $V_{k,a_p,\chi} \leftrightarrow \Pi_{k,a_p,\chi}$ in the crystalline case. In the next lecture, we will go to the general case and see that it's far much more complicated and doesn't lead to such an explicit $1-1$ correspondence ...

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