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Advanced Lecture Course
Lecture 1: How to use this 2-weeks conference

2010 Program for Women and Mathematics

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Introduction

The Langlands program, containing the so-called Langlands correspondences, take place in two different settings:

- the classical case, which is the ℓ-adic case;
- the non-classical case, which is the p-adic case (and is still a wide open question).

But what is a Langlands correspondence?

It's a bijection between two isomorphism classes:

$$\left\{ \begin{array}{c} \text{some "automorphic"} \\ \text{representations} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{some Galois} \\ \text{representations} \end{array} \right\} . \tag{1}$$

The left hand side of (1) will be called the $\mathbf{GL_n}(\mathbb{Q}_\mathbf{p})$ side; it will be treated during the second week of the conference in R. Ollivier and M.-F. Vignéras lectures, and in K. Körner and A. Cariani homework sessions.

The right hand side of (1) will be called the **Galois** side; it will be treated during this first week in E. Mantovan and A. Mézard lectures, and in L. Peskin and R. Abdellatif homework sessions.

Note that to explain how to get the correspondence (1), we will need to introduce a third object, namely

$$\{ \text{ unitary } G\text{-Banach spaces } \} \tag{2}$$

where G will be some $GL_n(\mathbb{Q}_p)$.

1 On the $GL_n(\mathbb{Q}_p)$ side

1.1 Notation

The field $\mathbb{Q}$ of rational numbers can be completed:

- either in a topological way, which leads to the $GL_n(\mathbb{Q}_p)$ side;
- or in an algebraic way, which leads to the Galois side.

As a consequence, the correspondence (1) can be seen as a way to link topology and algebra.

About the topological completion:

If we fix a prime number p, any $\alpha \in \mathbb{Q}^\times$ can be written $\alpha = \frac{m}{n}p^s$ with $s \in \mathbb{Z}$ and m, n integers such that $\gcd(n,p) = \gcd(m,p) = \gcd(m,n) = 1$. We then set

$$|\alpha|_p = p^{-s} .$$

This defines an absolute value $|.|_p$ over $\mathbb{Q}$, and we denote by $\mathbb{Q}_p$ the completion of $\mathbb{Q}$ for $|.|_p$. This absolute value can be extended to $\mathbb{Q}_p$, turning it (and also

$GL_n(\mathbb{Q}_p)$) into a topological space. For a more detailed explanation and for proofs, we refer to E. Mantovan lectures.

We are interested in $GL_n(\mathbb{Q}_p)$-representations, which are group homomorphisms

$$\pi : GL_n(\mathbb{Q}_p) \longrightarrow \mathrm{Aut}_{\mathbb{Q}_\ell}(V)$$

where ℓ is some prime and V is a $\mathbb{Q}_\ell$-vector space (maybe of infinite dimension), or even an E-vector space for some finite extension E of $\mathbb{Q}_\ell$. Such a π is called

- an *ℓ-adic representation* if $\ell \neq p$;
- a *p-adic representation* if $\ell = p$.

Note that there are really MANY MORE p-adic representations, what will bring some difficulties later...

1.2 The ℓ-adic case

Fix a prime ℓ different from p. In this setting, the so-called "automorphic representations[1]" that we consider in (1) are precisely the following ones:

$$\left\{ \textbf{irreducible smooth integral } GL_n(\mathbb{Q}_p)\text{-representations } \textbf{over } \overline{\mathbb{Q}_\ell} \right\} . \quad (3)$$

We are using the following terminology:

- **over** $\overline{\mathbb{Q}}_\ell$ means that the representations are defined over E-vector spaces with E some finite extension of $\mathbb{Q}_\ell$ contained in $\overline{\mathbb{Q}}_\ell$ and depending on the representation;
- **irreducible** means that they contain no non-trivial subrepresentation of $GL_n(\mathbb{Q}_p)$;
- **smooth** means that for any vector v of such a representation π, the set $\{g \in GL_n(\mathbb{Q}_p) |\ \pi(g)v = v\}$ is an open subgroup of $GL_n(\mathbb{Q}_p)$;
- **integral** means that there exists a $GL_n(\mathbb{Q}_p)$-invariant norm $||.||$ over V, i.e. a norm $||.||$ such that:

$$\forall v \in V,\ \forall g \in GL_n(\mathbb{Q}_p),\ ||gv|| = ||v|| .$$

Now we define what we called "unitary G-Banach spaces" in (2). They are the following objects:

$$\left\{ \text{topological absolutely irreducible unitary } GL_n(\mathbb{Q}_p)\text{-Banach spaces over } \overline{\mathbb{Q}}_\ell \right\} .$$

An element Π of this set is a Banach space endowed with a continuous linear E-action of $G := GL_n(\mathbb{Q}_p)$ such that the topology on Π is given by a G-invariant norm.

Vignéras [V] proved the following theorem:

[1] even if they are not automorphic in the sense that will appear in Section 3.1.

Theorem 1. *There is an equivalence of categories between $GL_n(\mathbb{Q}_p)$-representations and unitary G-Banach spaces defined as follows:*

- *to a unitary G-Banach space Π is associated the $GL_n(\mathbb{Q}_p)$-representation defined on the space Π^{sm} of its smooth vectors;*
- *to a $GL_n(\mathbb{Q}_p)$-representation π is associated the unitary G-Banach space defined by the completion $\widehat{\pi}$ of π for the topology induced by some norm satisfying the integrality property in (3).*

This theorem implies in particular that all the G-invariant norms on a given representation of $GL_n(\mathbb{Q}_p)$ induce the same topology on this representation.

1.3 The p-adic case

As we already mentionned it, we really have too many representations to deal with in the p-adic case, so that Theorem 1 is now false. To try to have something similar, we have to add some extra hypotheses:

- For the representations of $GL_n(\mathbb{Q}_p)$, we replace the smoothness hypothesis by the two following conditions:
  - ◇ **local analycity**: this means that for any $v \in \pi$, the map $[g \mapsto gv]$ is locally analytic (i.e. any small enough open compact subgroup of G acts analytically on v);
  - ◇ **strong admissibility**: this notion is due to Schneider-Teitelbaum [ST] and won't be defined here. It may be defined in next week's lectures.
- For the G-Banach spaces, we add an hypothesis of **admissibility**.

Following this way, we can still define a map $[\Pi \mapsto \Pi^{\text{loc an}}]$ from the $GL_n(\mathbb{Q}_p)$-representations to the G-Banach spaces, but there exists no reverse map as before: there are many G-invariants norms on a given π that will give rise to many different topologies (and so to many different Banach spaces associated to the same π).

2 On the Galois side

2.1 Definitions

Another way to complete $\mathbb{Q}$ is to consider its algebraic closure $\overline{\mathbb{Q}}$. This algebraically closed field is very difficult to understand: a lot of questions about it are still open questions. To try to obtain some informations on it, we study the associated Galois group $G_{\mathbb{Q}} := \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. This means that we study the representations of $G_{\mathbb{Q}}$, i.e. the group homomorphisms

$$\rho : G_{\mathbb{Q}} \longrightarrow GL_2(E)$$

with E being a finite extension of $\mathbb{Q}_\ell$. Note that we only consider here GL_2 (and not any GL_n) as this is the only case where we have some results in the

p-adic setting. For any prime p, such a representation ρ defines by restriction a representation of $G_{\mathbb{Q}_p} := \mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$:

$$\rho_p : G_{\mathbb{Q}_p} \longrightarrow GL_2(E) \ .$$

This representation will be called *ℓ-adic* when $\ell \neq p$ and *p-adic* when $\ell = p$.
In the ℓ-adic case, all representations of $G_{\mathbb{Q}_p}$ are well known as we can control the ℓ-part of $G_{\mathbb{Q}_p}$. In the p-adic case, there are once again so MANY representations...

2.2 The p-adic case

To manage to deal with the representations we are working on, we have to add some extra assumptions. Fontaine's idea was to first restrict to some "geometric" Galois representations (coming by example from the modular forms, or from étale cohomology). This leads in particular to two families of Galois representations:

$$\{ \text{ crystalline representations } \} \subset \{ \text{ semi-stable representations } \} \ .$$

These representations are much easier to understand: up to isomorphism, they are parametrized by a pair of parameters $(k, \mathcal{L}) \in \mathbb{N} \times \overline{\mathbb{Q}}_p$.

Theorem 2 (Colmez-Fontaine)**.** *There is an equivalence of categories:*

$$\left\{ \begin{array}{c} \textit{semi-stable representations} \\ G_{\mathbb{Q}_p} \longrightarrow GL_2(\overline{\mathbb{Q}}_p) \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \textit{admissible filtered } (\varphi, N)\textit{-modules} \\ \textit{of dimension } 2 \end{array} \right\} .$$

To have more explanations about this result, we refer to the Lecture 2 where we will study crystalline representations (which form a subcategory of the semi-stable representations).
We can nevertheless already give a rough definition: a **filtered** (φ, N)**-module** is a $\overline{\mathbb{Q}}_p$-vector space $D := \overline{\mathbb{Q}}_p e_1 + \overline{\mathbb{Q}}_p e_2$ of dimension 2 endowed with a Frobenius operator φ which is bijective, a monodromy operator N which is nilpotent, a filtration parametrized by a pair $(k, \mathcal{L})$ and some conditions on these three objects.
Note that crystalline representations will correspond to a monodromy $N \equiv 0$, what makes them easier to study.

3 How do relate these two sides?

3.1 The ℓ-adic case

We want a bijection of the following form:

$$\left\{ \begin{array}{c} \text{integral smooth irreducible} \\ \text{representations of } GL_n(\mathbb{Q}_p) \text{ over } \overline{\mathbb{Q}}_\ell \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \text{irreducible } \ell\text{-adic representations} \\ \rho_\ell : G_{\mathbb{Q}_p} \longrightarrow GL_n(\overline{\mathbb{Q}}_\ell) \end{array} \right\} .$$

It has been proved to be true for all $n \geq 2$ by Harris-Taylor (1998) and Henniart (with a simpler proof, 2000). Some other results have already been obtained earlier:

- Laumon-Rapoport-Stuhler (1993, [LRS]): for local fields of characteristic p (instead of $\mathbb{Q}_p$);
- Drinfeld (1989, [D]): for global function fields and $n = 2$;
- Lafforgue (2000, [L]): for global function fields and n arbitrary.

The idea of Harris-Taylor's proof is to work with global objects, i.e. to consider all prime numbers at the same time:

- on the Galois side, they consider $G_{\mathbb{Q}}$ and conclude by restriction;
- on the GL_n-side, they consider automorphic representations, i.e. representations of $GL_n(\mathbb{A})$ where $\mathbb{A}$ denotes the **adèle ring**, defined as the restricted product subring in $\mathbb{R} \times \prod_{\ell} \mathbb{Q}_\ell$.

3.2 The p-adic case

On the one hand, the correspondence in the ℓ-adic case gives a natural candidate for a p-adic correspondence :

$$\left\{ \begin{array}{c} \text{irreducible semi-stable} \\ \rho_p : G_{\mathbb{Q}_p} \to GL_2(\overline{\mathbb{Q}}_p) \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \text{admissible irreducible representations} \\ \text{of } GL_2(\mathbb{Q}_p) \text{ over } \overline{\mathbb{Q}}_p \end{array} \right\}$$
$$\rho_p \mapsto \pi_p$$

where π_p is given by the global construction of Harris-Taylor.
On the other hand, Colmez-Fontaine Theorem says that the left-hand side of this map is equivalent to the category of admissible filtered (φ, N)-modules. It seems and it had been proved in some particular cases (but not in general) that the π_p's could more or less be rebuilt from these objects. This would imply that ρ_p couldn't be rebuilt from π_p as the parameters $(k, \mathcal{L})$ will be missing !

Then came Breuil [Br] and his wonderful ideas to keep k and $\mathcal{L}$ in the construction:

- To keep k, we consider the following representation:
$$\underline{Sym}^{k-2}(\overline{\mathbb{Q}}_p^2) \otimes \pi_p \ . \tag{4}$$
Here $\underline{Sym}^{k-2}(\overline{\mathbb{Q}}_p^2)$ is the representation of $GL_2(\mathbb{Q}_p)$ defined as follows:
  - ◇ we start from the representation $Sym^{k-2}(\overline{\mathbb{Q}}_p^2)$ of $GL_2(\mathbb{Z}_p)$ whose underlying space is $\bigoplus_{i=0}^{k-2} \overline{\mathbb{Q}}_p u^i$ on which $GL_2(\mathbb{Z}_p)$ acts by
  $$\begin{pmatrix} a & b \\ c & d \end{pmatrix} u^i = (au + c)^i (bu + d)^{k-2-i} \ ;$$
  - ◇ we extend it to a representation of $GL_2(\mathbb{Z}_p)\mathbb{Z}_p^\times$ by sending p to the identity, and then to a representation of $GL_2(\mathbb{Q}_p)$ by compact induction (Advanced Homework Session 1).

- To keep $\mathcal{L}$, we add some topological structure by considering the completion of (4) for a norm depending on $\mathcal{L}$. This defines a topologically irreducible admissible G-Banach space $B(k, \mathcal{L})$.

  Conjecture 1 (Breuil, 2004). *If ρ_p is semi-stable and absolutely irreducible, then $B(k, \mathcal{L})$ is an object of (2). It determines ρ_p and only depends on it.*

This conjecture is now a theorem which has been proved by different methods:

- Breuil-Mézard [BM02, BM10], made some mod p computations, which gave the theorem for $4 \leq k \leq p+1$ even and $v_p(\mathcal{L}) \geq 0$;
- Breuil-Emerton [BE] built a geometric realization of the correspondence between ρ_p and the dual space of $B(k, \mathcal{L})$ when ρ_p is attached to a modular form;
- Berger-Breuil [BB10] and Colmez [C] improved Fontaine's theory of (φ, Γ)-modules and built a Borel-equivariant model of $B(k, \mathcal{L})^\vee$. The point is that this method gives the correspondence for *any* absolutely irreducible p-adic Galois representations of dimension 2.

4 Description of the next lectures and references

4.1 Description of the next lectures

- In Lecture 2, we study the Langlands correspondence for crystalline representations, which form a subcategory of the Galois representations we are interested in. We consider this particular case because everything is then rather explicit, so that we manage to get a very explicit correspondence for crystalline representations.

- In Lecture 3, we give some of the ideas that are needed to deal with the general case. In particular, we roughly explain how the theory of (φ, Γ)-modules is related to the Langlands correspondence for modulo p Galois representations and to the characteristic 0 setting, that will be studied in M.-F. Vignéras lectures.

- In Lecture 4, we discuss some variations and open questions on these topics.

4.2 Some references

This set of four lectures is just a short introduction to the p-adic Langlands program. To have more precise references, we recommand to check the following websites, as they contain complete lectures about some of the topics that will be discussed during this week:

- L. Berger: http://www.umpa.ens-lyon.fr/~lberger/ ;
- Ch. Breuil: http://www.ihes.fr/~breuil/ ;

- P. Colmez: http://www.math.jussieu.fr/~colmez/ ;
- A. Mézard: http://www.math.uvsq.fr/~mezard/ (especially for the notes entitled "Introduction à la correspondance de Langlands locale").

Références

[BB04] L. Berger, Ch. Breuil, *Towards a Langlands correspondence*, Notes of a course given at Hangzhou C.M.S (2004).

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[BM10] Ch. Breuil, A. Mézard, *Représentations semi-stables de* $GL_2(\mathbb{Q}_p)$*, demi-plan* p*-adique et réduction modulo* p, Astérisque 331 (2010) 117–178.

[C] P. Colmez, *Représentations de* $GL_2(\mathbb{Q}_p)$ *et* (φ, Γ)*-modules*, Astérisque 330 (2010), 281–509.

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[ST] P. Schneider, J. Teitelbaum, *Locally analytic distributions and* p*-adic representation theory, with application to* GL_2, J. Amer. Math. Soc. 15 (2002), 443–468.

[V] M.-F. Vignéras, ℓ*-adic Banach continuous representations of reductive* p*-adic groups when* $\ell \neq p$, preprint (2005).