

# IAS: 2019 SUMMER COLLABORATORS RESEARCH REPORT

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ABSTRACT. We investigated algebro-geometric, topological, and combinatorial structures of Springer fibers for nilpotent elements with at most three Jordan blocks in connection with the representation theory of the symmetric group and the Hecke algebra (e.g. Specht modules, Kazhdan–Lusztig bases, etc.).

## 1. SPRINGER FIBERS, KAZHDAN–LUSZTIG BASES AND SPECHT MODULES

Let  $x$  be an element in the nilpotent cone  $\mathcal{N} \subseteq \mathfrak{gl}_n(\mathbb{C})$ , and let  $\mu : \tilde{\mathcal{N}} \rightarrow \mathcal{N}$  be the Springer resolution. The study of Springer fibers  $\mu^{-1}(x)$  is closely related to studying Specht modules for the symmetric group. More precisely, the top non-vanishing cohomology  $H^{\text{top}}(\mu^{-1}(x))$  of the Springer fiber (equipped with the Springer action [Spr78]) is isomorphic to the irreducible Specht module labeled by the partition which encodes the Jordan type of  $x$ . Specht modules have a purely combinatorial basis, called Specht basis, naturally labeled by certain standard Young tableaux. Thus, it is a natural question to compare the Specht basis and the basis of  $H^{\text{top}}(\mu^{-1}(x))$  given by the classes of the irreducible components of the Springer fiber. For  $x$  with two Jordan blocks, it is known that geometric/topological structures of Springer fibers can be captured completely and explicitly using combinatorial tools, such as cup diagrams, standard Young tableaux (or partitions) of two rows. This fact makes the comparison of the two bases easily accessible from a combinatorial point of view.

For the two-row case of equal size, Russell–Tymoczko [RT17] compare the Specht basis with another combinatorial basis (called the web basis), which arises in the study of Temperley–Lieb algebras and in knot theory. In this case, the web basis gives a diagrammatic description of the Kazhdan–Lusztig basis, but this is not true in general. Moreover, in the two-row case, the web basis coincides with the basis of the top cohomology given by the classes of the irreducible components of  $\mu^{-1}(x)$ . The main result of [RT17] is a combinatorial model that gives a different proof to a special case of a classical theorem by Naruse [Nar89, Theorem 4.1] that the change-of-basis matrix between the Specht basis and the Kazhdan–Lusztig basis is unitriangular, altogether with some vanishing conditions (the unitriangularity result can also be found in [GM88], without the vanishing conditions). They also formulated a conjecture on the positivity of the coefficients, which was partially proved by [Rho18] as a non-negativity statement without a precise vanishing condition. We prove the positivity conjecture completely, with a precise vanishing condition.

On the other hand, we studied this problem using a different approach. Using the Kazhdan–Lusztig theory for the parabolic Hecke modules, we give a much shorter proof of [Nar89, Theorem 4.1], which includes the main theorems in [RT17] as a special case.

We further notice that our argument applies to all classical types if we adapt a nonstandard notion of tableaux. For example, in type B = C we use certain centro-symmetric tableaux which are not the bi-tableaux that parametrize the irreducibles. That is to say, the analogue of the Specht modules here correspond to the top cohomology of the Springer fibers with respect to certain parabolic subgroups, and hence they are in general not irreducible. For type D, we use the same Young diagrams as in type B/C, but with an extra restriction on the entries.

## 2. THREE-ROW SPRINGER FIBERS: GEOMETRIC AND TOPOLOGICAL STRUCTURES

The second week of our visit was spent on studying three-row Springer fibers, discussing existing web diagrams, aiming to redefine webs that reflect the geometry and topology of these Springer fibers. We start with the special case when  $x$  is a nilpotent element of Jordan type  $(m, m, m)$ . In the spirit of [Kho04, Weh09, RT11, Rus11], we aim to construct a topological three-row Springer fiber  $\mathcal{S}_{m,m,m}^{\text{top}}$  together with a homeomorphism

$$\mu^{-1}(x) \simeq \mathcal{S}_{m,m,m}^{\text{top}} = \bigcup_{\mathbf{a} \in \mathbb{B}_{m,m,m}} S_{\mathbf{a}},$$

where  $\mathbf{a}$  runs over a subset  $\mathbb{B}_{m,m,m}$  of certain  $\mathfrak{sl}_3$ -webs, and each  $S_{\mathbf{a}}$  is an explicit subvariety of  $(\mathbb{CP}^2)^{3m}$  defined by the web  $\mathbf{a}$ .

We then use various techniques from geometric invariant theory [MFK94, New78] to study algebro-geometric and topological properties of Springer fibers for three-row partitions, making connections to certain quasi-projective subvarieties in the cotangent bundle of the flag variety, which are in some sense adapted to a fixed nilpotent. For the two-row case (not necessarily of equal size), we have verified, for several small rank cases, a conjecture regarding certain favorable properties of these Springer fibers. A potential proof may be given by induction on the number of boxes on the second row of a Young tableau using techniques from [CK08]. We have completed the induction step, and certain details remain to be verified for the most general  $(n-1, 1)$ -partition. Our next step in this project includes investigating the patterns we have observed for the three-row case.

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