COLLABORATION SUMMARY: ALGEBRAIC AND SYMPLECTIC GROMOV-WITTEN THEORY

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The purpose of this collaboration is to provide an accessible comparison between algebraic and symplectic Gromov-Witten theory for closed curves, with the ultimate aim of showing that the two approaches give the same counts of curves in a smooth projective manifold. We define the algebraic invariants via the Behrend–Fantechi approach [BF] using algebraic stacks, while we take the symplectic invariants to be defined using the Hofer–Wysocki–Zehnder [HWZ2] theory of polyfolds. This project was started during the MSRI program on enumerative geometry in the Spring of 2018, and we envisage it taking a considerable time to complete, since the techniques used to define the two different versions of the invariants are so different.

As a first test case, we aim to establish a symplectic version of the genus zero Lefschetz hyperplane principle:

If $V \subset \mathbb{CP}^{n+1}$ is a smooth algebraic hypersurface, the genus zero algebraic GW-invariants for degree d curves in V are given by the Euler class of an orbibundle $\mathcal{H}^{alg} \to \mathcal{M}^{alg}$ over the finite dimensional orbifold of degree d holomorphic stable maps to $\mathbb{CP}^{n+1}$. They agree with the symplectic GW-invariants $\lim_{p\to 0}[(\mathfrak{s}+p)^{-1}(0)]$ obtained by perturbing the Cauchy-Riemann section $\mathfrak{s}_V^{HWZ} : \mathcal{X}_V \to \mathcal{W}_V$ over the polyfold $\mathcal{X}$ of stable maps to V from [HWZ1].

Even in this simple case, the smooth structure on $\mathcal{M}^{alg}$ is not the same as the smooth structure on $\mathcal{M}^{HWZ} = \mathfrak{s}_{\mathbb{CP}^{n+1}}^{HWZ\,-1}(0) \subset \mathcal{X}_{\mathbb{CP}^1}$ provided by the polyfold approach. And while there is a bijective smooth map $\psi : \mathcal{M}^{HWZ} \to \mathcal{M}^{alg}$, the pullback bundle $\psi^*\mathcal{H}^{alg} \to \mathcal{M}^{HWZ}$ has no obvious extension to the polyfold $\mathcal{X}_{\mathbb{CP}^1}$ of stable (nonholomorphic) maps to $\mathbb{CP}^1$.

Once the Lefschetz case is complete, the next aim is to tackle the general genus zero case. Finally, we will consider curves of arbitrary genus, where there is the new challenge of dealing with higher genus ghost (i.e. constant) components.

Our short visit to IAS during July 13 - 17, 2019 was very productive, and we are very grateful to IAS for providing us with this opportunity to get together.

- One main task is to enable the very different languages used in the two approaches to talk to one another. The face to face meetings were very helpful in sorting out the nuances of the two languages, and in understanding which features/constructions in the algebraic theory might carry over to an (sc-)smooth context.
- In 2018 we had formulated a finite-dimensional local model for combining symplectic and algebraic local charts near a non-nodal curve. This involved embeddings into a common ambient space and a homotopy. While at IAS we were able to simplify it to embeddings of both the symplectic and algebraic charts into a common comparison chart.
- We moreover sketched a proof outline for establishing the Lefschetz theorem by studying the compatibility between the different local comparision charts, and extending these constructions to neighbourhoods of nodal curves.

Based on our work at IAS, we formulated the following tasks for the coming year. Of these, 1,4 and 5 are needed for the Lefschetz project, while 2 and 3 look beyond it.

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1. Fantechi plans to write one or more short papers developing a theory of C^∞-stacks. As part of this, she will develop a functorial understanding of the change of smooth structure (for example on Deligne–Mumford space, or the Artin stack of stable curves) involved in going from the algebraic to the polyfold setting.

2. Fantechi will also develop a construction that embeds a neighbourhood of a given holomorphic genus zero stable map $[u : \Sigma \to V]$ (where the target manifold V is projective of complex dimension n) into an open set of regular stable maps in $\mathbb{C}P^{n+1}$. It seems very likely that such a construction exists, but if so, we need to understand its properties.

3. Fantechi and McDuff will investigate the normal cone construction, with a view to constructing a local geometric cycle that represents this cone in a homology theory that makes sense in the context of the finite dimensional models constructed in 4 and 5 below.

4. McDuff and Wehrheim will complete their paper [MW2] that constructs finite dimensional models of sc-Fredholm sections of polyfold bundles which preserve the virtual fundamental class. These models include Kuranishi atlases as in [MW1], as well as a new finite dimensional model consisting of a trivialized bundle over a branched (nonHausdorff/nonseparated) manifold, together with a global group action as in [M].

5. To establish the Lefschetz theorem, we aim to extend our construction of local comparision charts to a Kuranishi atlas as defined in [MW1]. Then we expect embedding equivalences between this atlas and the bundle $\mathcal{H}^{alg} \to \mathcal{M}^{alg}$, as well as the Kuranishi atlas resulting from applying [MW2] to the polyfold section $\mathfrak{s}_V^{HWZ} : \mathcal{X}_V \to \mathcal{W}_V$.

References

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