# 2019 IAS Summer Collaborators Program

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## Spectral Dynamic Analysis of Stochastic Service Networks

**Free Process:** All the state spaces for the random processes that we consider are all subsets of $\mathbb{Z}^d$, the *integer lattice* of dimension $d$.

$$\mathbb{Z}^d \equiv \left\{ [n_1, \ldots, n_d] \,\middle|\, n_j \in \mathbb{Z} \text{ for all } j = 1, \ldots, d \right\} = \left\{ \sum_{j=1}^{d} n_j \cdot \mathbf{e}_j \,\middle|\, n_j \in \mathbb{Z} \text{ for all } j = 1, \ldots, d \right\},$$

where $\mathbf{e}_j$ is the $j$ - th *unit vector* for $j = 1, \ldots, d$.

Now assume the following three sets of objects:

1. We have $d+1$ independent standard (rate 1) Poisson processes $\left\{ \Pi_j(t) \,\middle|\, t \geq 0 \right\}$ where $j = 0, 1, \ldots, d$.
2. We have $d+1$ vectors $\left\{ \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_d \right\} \subseteq \mathbb{Z}^d$ where $\sum_{j=0}^{d} \mathbf{v}_j = 0$, all subsets with $d$ elements are linearly independent, and
$$\mathbb{Z}^d = \left\{ \sum_{j=0}^{d} n_j^* \cdot \mathbf{v}_j \,\middle|\, n_j^* \in \mathbb{Z} \text{ for all } j = 1, \ldots, d \right\}.$$
3. We have $d+1$ positive real constants $\mu_0, \mu_1, \ldots, \mu_d$.

We define the Markov process $\mathbf{Z} \equiv \left\{ \mathbf{Z}(t) \,\middle|\, t \geq 0 \right\}$ on $\mathbb{Z}^d$ to be a "randomized random walk" where:

$$\mathbf{Z}(t) = \mathbf{Z}(0) + \sum_{j=0}^{d} \Pi_j(\mu_j \cdot t) \cdot \mathbf{v}_j.$$

Each $\left\{ \Pi_j(\mu_j t) \,\middle|\, t \geq 0 \right\}$ is a Poisson process with rate $\mu_j$. Our random walk process is an *irreducible* Markov process whenever the positive (including zero) integer span of the vectors $\left\{ \mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_d \right\}$ is the entire lattice $\mathbb{Z}^d$. Hence, there can be no fewer than $d+1$ vectors. This number is sufficient only if deleting any single vector leaves the integer span of the remaining set vectors equaling the entire lattice. We define these vectors to be a *projective basis* for $\mathbb{Z}^d$.

The resulting Markov process $\mathbf{Z}$, is the simplest irreducible randomized random walk on $\mathbb{Z}^d$. Two different projective bases are equivalent up to a lattice automorphism (integer matrices with determinant $1$ or $-1$). Hence, we can use the following projective basis for $\mathbb{Z}^d$ with no loss of generality:

$$\begin{aligned}\mathbf{v}_0 &\equiv \begin{bmatrix}1 & 0 & 0 & \cdots & 0\end{bmatrix} = \mathbf{e}_1 \\ \mathbf{v}_1 &\equiv \begin{bmatrix}0 & -1 & 1 & \cdots & 0\end{bmatrix} = \mathbf{e}_2 - \mathbf{e}_1 \\ &\vdots \\ \mathbf{v}_{d-1} &\equiv \begin{bmatrix}0 & \cdots & 0 & -1 & 1\end{bmatrix} = \mathbf{e}_d - \mathbf{e}_{d-1} \\ \mathbf{v}_d &\equiv \begin{bmatrix}0 & \cdots & 0 & 0 & -1\end{bmatrix} = -\mathbf{e}_d\end{aligned}$$

Notice that this is equivalent to defining $\mathbf{e}_j \equiv \mathbf{v}_0 + \mathbf{v}_1 + \cdots + \mathbf{v}_{j-1}$.

The one dimensional case for this problem was addressed in [Feller – 1971], where he coined the phrase randomized random walk. Using the operation research motivation of public transportation services, this free process models a taxi service station, where the Poisson process at rate $\mu_0 \equiv \lambda$ models the customer arrival count during a fixed time interval and the Poisson process at rate $\mu_1 \equiv \mu$ models the corresponding taxi arrival count. A strictly positive integer state $[n]$ models the number of customers waiting for a taxi at the station but a strictly negative integer state $[n]$ models the number of taxicabs waiting for a customer.

Since $\mathbf{v}_0 = [1]$ and $\mathbf{v}_1 = [-1]$, the event of an arriving customer is modelled as a state transition $[n] \to [n+1]$. This happens at a Poisson rate $\lambda$. We also model the event of an arriving taxi as a state transition $[n] \to [n-1]$. This event happens at a Poisson rate $\mu$. A taxi can then be viewed as a "negative customer" or an "anti-customer".

The one-dimensional version of our free process is then

$$\mathrm{Z}(t) = \mathrm{Z}(0) + \Pi_0(\lambda \cdot t) - \Pi_1(\mu \cdot t).$$

In [Feller – 1971] the transition probabilities for this free process are solved exactly in terms of modified Bessel functions where

$$\mathrm{P}_m\left\{Z(t)=n\right\} \equiv \mathrm{P}\left\{Z(t)=n \middle| Z(0)=m\right\} = e^{-(\lambda+\mu)t} \cdot \sqrt{\frac{\lambda}{\mu}}^{\,n-m} \cdot I_{n-m}\left(2t\sqrt{\lambda\mu}\right).$$

This solution can be expressed more simply in terms of trigonometric functions as

$$\mathrm{P}_m\left\{Z(t)=n\right\} \equiv \frac{\sqrt{\lambda/\mu}^{\,n-m}}{2\pi} \cdot \int_{-\pi}^{\pi} e^{-\left(\lambda+\mu-2\sqrt{\lambda\mu}\cos\theta\right)t} \cdot \cos\left(m-n\right)\theta \, d\theta\,.$$

A new class of Bessel functions that expresses a solution for the transition probabilities of the multi-dimensional free process was created in [Massey – 1987].

*This summer*: We have developed new techniques for deriving the transition probabilities that bypass the use of these Bessel functions or transform methods such as generating functions or Laplace transforms. The key tools are probabilistic in nature. We ultimately exploit the natural group symmetries of the process and use complex contour integration. We can then directly transform the random process into a multiple complex integral that expresses its transition probabilities as

$$\mathrm{P}_{\mathbf{m}}\left\{\mathbf{Z}(t)=\mathbf{n}\right\} = \frac{\boldsymbol{\beta}^{\mathbf{n}-\mathbf{m}}}{\left(2\pi i\right)^d} \oint_{|w_1|=r_1} \cdots \oint_{|w_d|=r_d} e^{-(d+1)\cdot(\alpha-\gamma\cdot\delta(\mathbf{w}))t} \cdot \mathbf{w}^{\mathbf{m}-\mathbf{n}} \frac{dw_1}{w_1} \wedge \cdots \wedge \frac{dw_d}{w_d},$$

for all $\mathbb{Z}^d$ lattice points $\mathbf{m} \equiv [m_1, \ldots, m_d]$ and $\mathbf{n} \equiv [n_1, \ldots, n_d]$. An optimal choice of contours is $r_1 = \cdots = r_d = 1$. This transforms this integral over the unit complex torus in $\mathbb{C}^d$ into the following real multiple integral over $\mathbb{R}^d$:

$$\mathrm{P}_{\mathbf{m}}\{\mathbf{Z}(t) = \mathbf{n}\} = \frac{\boldsymbol{\beta}^{\mathbf{n}-\mathbf{m}}}{(2\pi)^d} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} e^{-(d+1)\cdot(\alpha - \gamma\cdot\delta(\boldsymbol{\theta}))t} \cdot e^{i\boldsymbol{\theta}\cdot\mathbf{m}} \cdot e^{-i\boldsymbol{\theta}\cdot\mathbf{n}} d\theta_1 \wedge \cdots \wedge d\theta_d,$$

where $i \equiv \sqrt{-1}$. This gives us the *spectral decomposition* of the Markov infinitesimal generator for the free process and asymptotic behavior of the transition probabilities as $t \to \infty$.

**Symmetry Group:** We can associate a natural symmetry group $G_d$ with this free process, where

$$G_d \equiv \left\{ \text{All linear operators on } \mathbb{Z}^d \text{ that permute the projective basis } \{\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_d\} \right\}.$$

With respect to a specific basis, the resulting matrices form a $d$ dimensional representation for the group of permutations on $d+1$ objects. For the one-dimensional case, with $\mathbf{v}_0 = [1]$ and $\mathbf{v}_1 = [-1]$, these group symmetry matrices reduce to multiplications by $1$ and $-1$, where $1 \cdot [n] = [n]$ and $-1 \cdot [n] = [-n]$.

Observe that for the one dimensional case, the positive integers $\mathbb{Z}_+$ (including zero) constitute a *fundamental domain* for the corresponding group action of $G_1$ on $\mathbb{Z}$. This is a minimal subset of $\mathbb{Z}$ whose group orbit equals $\mathbb{Z}$. Strictly positive numbers have an orbit of two points, namely $[n]$ and $[-n]$. The *boundary* $[0]$ of $\mathbb{Z}_+$ is uniquely characterized by being uniquely invariant with respect to this group action.

The transition probabilities inherit group symmetry properties. One type of *algebraic symmetry* is *translation invariance* or

$$\mathrm{P}_m\{Z(t) = n\} = \mathrm{P}_{m+\ell}\{Z(t) = n + \ell\}.$$

for all integers $\ell$. This follows from

$$\mathrm{Z}(t) = \mathrm{Z}(0) + \Pi_0(\lambda\cdot t) - \Pi_1(\mu\cdot t) \quad \Leftrightarrow \quad \mathrm{Z}(t) + \ell = \mathrm{Z}(0) + \ell + \Pi_0(\lambda\cdot t) - \Pi_1(\mu\cdot t).$$

Our queueing network state space, the positive orthant $\mathbb{Z}_+^d$, can be expressed in terms of our given projective basis as

$$\mathbb{Z}_+^d \equiv \left\{ [n_1, \ldots, n_d] \,\middle|\, \min(n_1, \ldots, n_d) \geq 0 \right\} \cap \mathbb{Z}^d = \left\{ \sum_{j=0}^{d} n_j^* \cdot \mathbf{v}_j \,\middle|\, n_j^* \in \mathbb{Z} \text{ for all } j = 1, \ldots, d \text{ where } n_0^* \geq n_1^* \geq \cdots \geq n_d^* \right\}.$$

This projective basis representation reveals that $\mathbb{Z}_+^d$ is a minimal subset of the lattice $\mathbb{Z}^d$, whose group orbit equals the entire lattice $\mathbb{Z}^d$. Hence $\mathbb{Z}_+^d$ is a fundamental domain on the lattice $\mathbb{Z}^d$ with respect to the group action $G_d$.

**Open Queueing Networks:** The simplest one dimensional open queueing network $Q \equiv \{Q(t) | t \geq 0\}$ is called an $M/M/1/\infty$ queue. The public transportation interpretation is a taxi cab station where no taxi stops unless they see a customer waiting. The state space is now $\mathbb{Z}_+$ and its Markovian transitions correspond to a reflecting version of the corresponding free process constrained to $\mathbb{Z}_+$. We can construct the sample paths for the $M/M/1/\infty$ queueing process out of the sample paths for the free process of the same rates by setting

$$Q(t) = Z(t) - \min\left(\inf_{0\le s\le t} Z(s), 0\right).$$

We define a service network to be *balanced* if all the stations are busy. The first time $T_0$ that this queueing system is no longer balanced is a stopping time for the $M/M/1/\infty$ queueing process. This stopping time is equivalent to the time until the absorption of the free process to the boundary, which we also call $T_0$.

$$\mathrm{P}_m\left\{Q(t)=n, T_0 > t\right\} = \mathrm{P}_m\left\{Z(t)=n, T_0>t\right\} = \frac{\sqrt{\lambda/\mu}^{\,n-m}}{\pi}\cdot\int_{-\pi}^{\pi} e^{-\left(\lambda+\mu-2\sqrt{\lambda\mu}\cos\theta\right)t}\cdot\sin m\theta\cdot\sin n\theta\, d\theta\,.$$

The simplest multi-dimensional open queueing network $\mathbf{Q} \equiv \left\{\mathbf{Q}(t) \middle| t \ge 0\right\}$ is called a *series open Jackson network*. Its state space is the set of positive vectors $\varDelta_d \equiv \mathbb{Z}_+^d$. Every vector $\mathbf{n} \equiv [n_1, \ldots, n_d]$, where $\min(n_1,\ldots,n_d) \ge 0,$ corresponds to the state of having $n_j$ customers at service station $j$ for all $j = 1,\ldots,d.$ The projective basis vectors correspond to the following set of queueing network transition events:

$$\begin{array}{lcl}
\mathbf{n}\to\mathbf{n}+\mathbf{v}_0 & \Leftrightarrow & \text{customer arrives to service station \#1.}\\
 & \vdots & \\
\mathbf{n}\to\mathbf{n}+\mathbf{v}_j & \Leftrightarrow & \text{customer transfers from service station \#}j\text{ to service station \#}j+1\ \text{ (when } 1\le j<d \text{ and } n_j>0).\\
 & \vdots & \\
\mathbf{n}\to\mathbf{n}+\mathbf{v}_d & \Leftrightarrow & \text{customer departs service station \#}d\ \text{ (when } n_d>0).
\end{array}$$

Markovian transitions of the queueing process $\mathbf{Q}$ correspond to a *reflecting* version of the multi-dimensional free process $\mathbf{Z}$ constrained to $\mathbb{Z}_+^d$. Jackson networks [Jackson – 1957] have the surprising feature of having correlated queueing stations that become independent in steady state. This means that whenever $\mu_0 < \min(\mu_1,\ldots,\mu_d)$ we then have

$$\lim_{t\to\infty}\mathrm{P}_{\mathbf{m}}\left\{\mathbf{Q}(t)=\mathbf{n}\right\} = \prod_{j=1}^{d}\left(1-\frac{\mu_0}{\mu_j}\right)\cdot\left(\frac{\mu_0}{\mu_j}\right)^{n_j}.$$

This complex integral for the transition probabilities of the queueing process with absorption has the form

$$\mathrm{P}_{\mathbf{m}}\left\{\mathbf{Z}(t)=\mathbf{n}, T_{\partial\mathbb{Z}_+^d} > t\right\} = \frac{\boldsymbol{\beta}^{\mathbf{n}-\mathbf{m}}}{(2\pi)^d}\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi} e^{-(d+1)\cdot(\alpha-\gamma\cdot\delta(\boldsymbol{\theta}))t}\cdot\varepsilon\left(\boldsymbol{\theta},\mathbf{m}\right)\cdot\varepsilon\left(\boldsymbol{\theta},\mathbf{n}\right)d\theta_1\wedge\cdots\wedge d\theta_d.$$

with

$$\mathrm{P}_{\mathbf{m}}\left\{\mathbf{Q}(t)=\mathbf{n}; T_{\partial\mathbb{Z}_+^d} > t\right\} = \mathrm{P}_{\mathbf{m}}\left\{\mathbf{Z}(t)=\mathbf{n}; T_{\partial\mathbb{Z}_+^d} > t\right\} \quad \text{where} \quad T_{\partial\mathbb{Z}_+^d} = \inf\left\{t \middle| \mathbf{Z}(t)\in\partial\mathbb{Z}_+^d\right\},$$

and

$$\partial\mathbb{Z}_+^d \equiv \left\{\left[n_1,\ldots,n_d\right] \middle|\ \min\left(n_1,\ldots,n_d\right)=0\ \right\}\cap\mathbb{Z}^d.$$

In [Massey – 1987], we obtain this solution directly by using the projective basis group symmetries to show that

$$\mathrm{P}_{\mathbf{m}}\left\{\mathbf{Q}(t)=\mathbf{n};T_{\partial\mathbb{Z}_+^d}>t\right\}=\sum_{g\in G_d}(-1)^g\,\boldsymbol{\beta}^{\mathbf{n}-g(\mathbf{n})}\cdot\mathrm{P}_{\mathbf{m}}\{\mathbf{Z}(t)=g(\mathbf{n})\}=\sum_{g\in G_d}(-1)^g\,\boldsymbol{\beta}^{g(\mathbf{m})-\mathbf{m}}\cdot\mathrm{P}_{g(\mathbf{m})}\{\mathbf{Z}(t)=\mathbf{n}\}$$

We can then complete the argument in three steps we show that:

1. At time $t=0$, the probabilities reduce to the value of $1$ if and only if $\mathbf{m}=\mathbf{n}$, otherwise they equal zero.
2. The sums satisfy the same Kolmogorov forward equations.
3. For all $t>0$, $\mathrm{P}_{\mathbf{m}}\left\{\mathbf{Q}(t)=\mathbf{n};T_{\partial\mathbb{Z}_+^d}>t\right\}=0$ whenever either $\mathbf{m}$ or $\mathbf{n}$ belong to the boundary $\partial\mathbb{Z}_+^d$.

To show that these three expressions are equivalent we also need to develop an important type of *probabilistic symmetry* for the free process. For the one-dimensional case, we have the following *reflection principle*

$$\mathrm{P}_0\left\{Z(t)=n\right\}=\left(\frac{\lambda}{\mu}\right)^n\cdot\mathrm{P}_0\left\{Z(t)=-n\right\}=\beta^{n-(-n)}\cdot\mathrm{P}_0\left\{Z(t)=-n\right\}\quad\text{where}\quad\beta\equiv\sqrt{\frac{\lambda}{\mu}}.$$

for all integer states $n$. This follows immediately from the symmetry properties of the modified Bessel functions and was shown in [Baccelli and Massey – 1989] from first principles by using stopping times and Laplace transforms. For the multi-dimensional case, it was shown in [Massey – 1987], by using lattice Bessel functions for open series Jackson networks, that there exists a strictly positive vector $\left[\beta_1,\ldots,\beta_d\right]$ in $\mathbb{R}^d$, such that

$$\mathrm{P}_{\mathbf{0}}\left\{\mathbf{Z}(t)=\mathbf{n}\right\}=\boldsymbol{\beta}^{\mathbf{n}-g(\mathbf{n})}\cdot\mathrm{P}_{\mathbf{0}}\left\{\mathbf{Z}(t)=g(\mathbf{n})\right\}.$$

for all $g\in G_d$ where $\boldsymbol{\beta}^{\mathbf{n}}\equiv\beta_1^{n_1}\cdot\beta_2^{n_2}\cdot\ldots\cdot\beta_d^{n_d}$.

*This summer*: We developed a simpler probabilistic proof of this result using a change of measure formula for independent Poisson distributions. Combining this with the translation symmetry gives us

$$\boldsymbol{\beta}^{\mathbf{m}-\mathbf{n}}\cdot\mathrm{P}_{\mathbf{m}}\left\{\mathbf{Z}(t)=\mathbf{n}\right\}=\boldsymbol{\beta}^{g(\mathbf{m})-g(\mathbf{n})}\cdot\mathrm{P}_{g(\mathbf{m})}\left\{\mathbf{Z}(t)=g(\mathbf{n})\right\}.$$

**Closed Cyclic Queueing Networks as Finite Capacity Open Series Networks:** The simplest closed queueing network is one that is *cyclic*. Since the total number in the system is a fixed number $k$, a closed cyclic network with $d+1$ service stations is equivalent to a open series network with $d$ stations, where new customer arrivals are blocked if and only if there are already $k$ customers in the system. For more details on its "product form" steady state distribution, see [Jackson – 1967].

A two node closed network with $k$ customers is always cyclic and is equivalent to an $M/M/1/k$ queueing station. This queueing system is a basic public transportation model for a bicycle sharing station having $k$ secured bicycle parking docks. One group of customers arrive at rate $\mu_0\equiv\lambda$ to rent an available bicycle. A second group of customers arrive at rate $\mu_1\equiv\mu$ to return a rented bicycle if an empty parking dock is available. Examples of current research on bike sharing can be found in [Schuijbroek, Hampshire, and Van Hoeve – 2017] and [Tao and Pender – 2017].

The state space here is restricted to $\boldsymbol{\Delta}_{1,k}\equiv\{0,1,\ldots,k\}$ instead of $\mathbb{Z}_+$. It has a *boundary* of two states where $\partial\boldsymbol{\Delta}_{1,k}=\{[0],[k]\}$. These are the two *unbalanced* states of customers approaching the station. One scenario is that they find no available bicycles, which is state $[0]$. The other case is they find no available empty parking docks for returning a bicycle, which is state $[k]$. Moreover, the state space $\boldsymbol{\Delta}_{1,k}$, as a subset of $\mathbb{Z}$, is the fundamental domain for the group $\mathcal{G}_{1,k}$ of reflections about these two boundary points. This group decomposes into a *semi-direct product* of the "rotations" of scalar multiplications by $G_1=\{1,-1\}$, combined with the "translations" of integer additions of even multiples of $k$, which are all the elements of $2k\mathbb{Z}$. Our symmetry group then corresponds to the set of "rigid motions" in Euclidean geometry. It can be shown that every element $g \in \mathcal{G}_{1,k}$ is then of the form $g(n) = n + 2k\ell$ or $g(n) = -n + 2k\ell$ for all integers $\ell$. We can summarize this as $\mathcal{G}_{1,k} = G_1 \times_s 2k\mathbb{Z}$.

For the multi-dimensional case, our state space is now $\Delta_{d,k}$ where

$$\Delta_{d,k} \equiv \left\{ [n_1,\ldots,n_d] \middle| \sum_{j=1}^{d} n_j \le k \text{ where } \min(n_1,\ldots,n_d) \ge 0 \right\} \cap \mathbb{Z}^d$$

We define $\partial\Delta_{d,k}$ to be the boundary of this state space where

$$\partial\Delta_{d,k} \equiv \left\{ [n_1,\ldots,n_d] \middle| \sum_{j=1}^{d} n_j = k \text{ or } \min(n_1,\ldots,n_d) = 0 \right\} \cap \mathbb{Z}^d.$$

We define the service network to be *unbalanced* when it equals one of these states. All these states describe the event of some service station in the closed network being empty.

This state space $\Delta_{d,k}$ is the fundamental domain for the group of “reflections” $\mathcal{G}_{d,k}$ acting on $\mathbb{Z}^d$. This group is the *semi-direct product* $\mathcal{G}_{d,k} = G_d \times_s \mathbf{\Lambda}_{d,k}$. Our group of “translations” $\mathbf{\Lambda}_{d,k} \subseteq \mathbb{Z}^d$ are defined to be $G_d$ - invariant sub-lattice of

$$\mathbf{\Lambda}_{d,k} \equiv \left\{ k \cdot \sum_{j=0}^{d} n_j^* \cdot \mathbf{v}_j \middle| n_j^* \in \mathbb{Z} \text{ for all } j = 1,\ldots,d \text{ with } \sum_{j=0}^{d} n_j^* = 0 \right\}.$$

The balance time for this closed or blocked network is a stopping time for this Markovian queueing process. It can then be computed from the absorption transition probabilities. In turn, we derive these by using group symmetry argument identical to the ones we used for the open network so we have:

$$\mathrm{P}_{\mathbf{m}}\left\{ \mathbf{Q}(t) = \mathbf{n}; T_{\partial\Delta_{d,k}} > t \right\} = \mathrm{P}_{\mathbf{m}}\left\{ \mathbf{Z}(t) = \mathbf{n}; T_{\partial\Delta_{d,k}} > t \right\} \quad \text{where} \quad T_{\partial\Delta_{d,k}} = \inf\left\{ t \middle| \mathbf{Z}(t) \in \partial\Delta_{d,k} \right\}$$

$$\Downarrow$$

$$\mathrm{P}_{\mathbf{m}}\left\{ \mathbf{Q}(t) = \mathbf{n}; T_{\partial\Delta_{d,k}} > t \right\} = \sum_{g \in \mathcal{G}_{d,k}} (-1)^g \boldsymbol{\beta}^{\mathbf{n} - g(\mathbf{n})} \cdot \mathrm{P}_{\mathbf{m}}\{\mathbf{Z}(t) = g(\mathbf{n})\} = \sum_{g \in \mathcal{G}_{d,k}} (-1)^g \boldsymbol{\beta}^{g(\mathbf{m}) - \mathbf{m}} \cdot \mathrm{P}_{g(\mathbf{m})}\{\mathbf{Z}(t) = \mathbf{n}\}.$$

The major difference from the open network is that we are now summing over an infinite group. The end result that is achieved through complex analysis is that these infinite sums collapse to a finite sum of exponentials. For more details on these symmetry issues, see [Baccelli, Massey, and Wright – 1994].

*This summer*: For the one dimensional finite capacity case of the $M/M/1/k$ queue, we combined group symmetry arguments with complex contour integration to develop simpler methods for deriving these transition probabilities for balanced states prior to being absorbed into one of the unbalanced states. Leaving out the contour integration steps, we have

$$
\begin{aligned}
\mathrm{P}_m\left\{Q(t)=n; T_{\partial\Delta_{1,k}}>t\right\} &= \sum_{g\in\mathcal{G}_{1,k}} (-1)^g \beta^{n-g(n)}\cdot \mathrm{P}_m\{Z(t)=g(n)\} \\
&= \sum_{\ell=-\infty}^{\infty} \beta^{n-(n+2k\ell)}\cdot \mathrm{P}_m\{Z(t)=n+2k\ell\} - \sum_{\ell=-\infty}^{\infty} \beta^{n-(-n+2k\ell)}\cdot \mathrm{P}_m\{Z(t)=-n+2k\ell\} \\
&= \sum_{\ell=-\infty}^{\infty} \left(\frac{\lambda}{\mu}\right)^{-k\ell}\cdot \mathrm{P}_m\{Z(t)=n+2k\ell\} - \sum_{\ell=-\infty}^{\infty} \left(\frac{\lambda}{\mu}\right)^{n-k\ell}\cdot \mathrm{P}_m\{Z(t)=-n+2k\ell\} \\
&= \quad \text{(contour integration steps)} \\
&= \frac{2\sqrt{\lambda/\mu}^{\,n-m}}{k}\cdot \sum_{j=1}^{k-1} \sin\frac{mj\pi}{k}\sin\frac{nj\pi}{k}\cdot e^{-\left(\lambda+\mu-2\sqrt{\lambda\mu}\cdot\cos\frac{j\pi}{k}\right)t}.
\end{aligned}
$$

Also, the context of a bicycle sharing station suggests an analysis of the expected transient balance time. For references, again see [Schuijbroek, Hampshire, and Van Hoeve – 2017] and [Tao and Pender – 2017]. We are now able to write an explicit formula for this expectation as a function of an initially balanced state. This can be used to find optimal starting load of bicycles that maximizes the mean balance time.

**Future Work:** We already have a complete spectral transient analysis for the free process and its behavior before absorption to the state space boundary. Our ultimate goal is to do the same for the reflecting version of these free processes. This would give us a complete spectral classification for the transient analysis of all fundamental open and closed queueing networks.

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