Progress report: the Laguerre-Pólya class, Pólya frequency sequences, and generalized Hurwitz matrices

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To begin with, let us recall that a real entire function ϕ belongs to the *Laguerre-Pólya class* ($\mathcal{L}-\mathcal{P}$) if and only if it admits the following representation

$$\phi(z) = Ce^{-az^2+bz}\, z^m \prod_{k=1}^{\omega}(1+z/z_k)e^{-z/z_k},$$

where $a \geq 0$, $m \in \mathbb{N}$, $\sum_k 1/|z_k|^2 < \infty$, and all z_k are all real (ω may be finite or infinite). An important subclass of this class, denoted $\mathcal{L}-\mathcal{P}^+$, consists of functions with only real negative zeros.

An interesting fact about the Laguerre-Pólya class is that one of the formulations of the Riemann Hypothesis is that the Riemann ξ-function is in $\mathcal{L}-\mathcal{P}^+$ [3].

Motivated by [1] we considered various interpolation problems for the Riemann ζ-function, which can then be translated and generalized to the entire Laguerre-Pólya class. In particular, we constructed the Thiele continued fraction based on the values of the Riemann ζ-function at negative integers, which are basically the Bernoulli numbers. Recall that a Thiele fraction is a continued fraction corresponding to an interpolation process [2].

In addition to a few theoretical attempts to understand the behavior of the entries of such Thiele fractions, we ran a bunch of numerical experiments and observed that Thiele fractions corresponding to different interpolation problems do not behave the way one would expect but we did not exhaust all possibilities. Still it would be interesting to keep investigating convergence properties of all of those continued fractions.

Another outcome of our discussions is that we will keep working on trying to find some universal properties of Thiele fractions for the Laguerre-Pólya class.

References

[1] K. Ball, *Rational approximations to the zeta function*, arXiv:1706.07998.

[2] A. Cuyt, V. Brevik Petersen, B. Verdonk, H. Waadeland, and W. B. Jones, *Handbook of Continued Fractions for Special Functions*, Springer, 2008.

[3] G. Pólya. *Collected papers.* The MIT Press, Cambridge, Mass.-London, 1974. Vol. II: Location of zeros, Edited by R. P. Boas, Mathematicians of Our Time, Vol. 8.